Basic Forms and Nature

From Visual Simplicity to Conceptual Complexity

Markus Rissanen

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Ulpulle ja Urmakselle

Preface

According to the rules of the Doctoral Studies Program of the Academy of Fine Arts, University of the Arts Helsinki, the candidate for the degree must give a public demonstration displaying a high level of skill, knowledge and research in their own field. After three pre-examined exhibitions, this dissertation completes my doctoral work.

When presenting my own paintings in this thesis, I have also given a continuously-running chronological number, which I have given to all of my paintings since my graduation from the Finnish Academy of Fine Arts in 2000. This number is in square brackets after the title of the work, for example: *Blueprints for Landscapes*, [M125]. The letter M can be taken to stand for *maalaus*, "painting" in Finnish.

Most of the material I refer to in this work is in printed form. At some point, however, I realized just how much printed material is already available online. Whenever I have noticed that a book or an article also has a free online version, I have provided the proper address. Where possible, I have favored a link where the context of the publication is also visible, even if access to the actual content requires a quick look and one extra click, over the alternative where the content just pops up out of the blue. Compare, for example, A. K. Dewney's article "Computer recreations; a computer microscope zoom in for a look at the most complex object in mathematics" in the August 1985 issue of the *Scientific American* at https://www.scientificamerican.com/article/mandelbrot-set/ within a clear and comprehensible context, versus exactly the same article at https://www.scientificamerican.com/media/inline/blog/File/Dewdney_Mandelbrot.pdf, with no explanatory context. With online sites, the access date is given as: (accessed 2017-05-20). If the available space in a footnote was especially tight, the access date was simply given as (2017-05-20). A descending notation for the dates (Year-Month-Day) is systematically used.

With all of my visits to such online resources, I often had the feeling of living in a world that was anticipated more than 70 years ago in an essay by William J. Wilson, "The Union Catalog of the Library of Congress", published in the journal ***Isis***, Vol. 33, No. 5, March 1942: "In the more distant future still stranger things may happen through television and its allied inventions. If these can be perfected for general use, as have the radio and the telephone, the results will be revolutionary indeed. While the facilities are lacking, already the principles are known by which any library could be equipped to show a rare book or a section of its card catalog to a distant scholar, sitting in his own study before his television screen and turning the pages or flipping the cards by remote control!" Nowadays all of this has become so familiar to us that we barely give it a second thought. For me, on several occasions, the kind of library just described has been the Internet Archive (https://archive.org).

The psychological roller-coaster ride caused by these doctoral studies, and especially this written part, is well expressed by Marjorie Hope Nicolson (1894–1981), an American scholar of 17th-century literature, in the title of her 1955 published book *Mountain Gloom and Mountain Glory*. For artistic research, this is indeed a suitable phrasing as being taken from the heads of chapters XIX and XX of the Part V of Volume 4 of the five-volume book *Modern Painters* (1843–60) by the Victorian art critic John Ruskin (1819–1900).

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The author can be reached via email at:
markus.rissanen@gmail.com

INTRODUCTION

In this chapter I present the topic and the research questions of this thesis. I also discuss the key concepts, such as the "basic forms" as well as the notions of figure, form, shape, structure, and pattern.

> *Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics and its characters are triangles, circles, and other geometrical figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.*
>
> Galileo Galilei, *Il Saggiatore* (1623)[1]

I am not a scientist, nor a philosopher, and the theme of this thesis is not words or dark labyrinths. As an artist doing research, I have studied basic forms, their characters and their presence in nature. The main forms, which I consider basic, are the square, circle, and triangle. In addition to these three simple forms, I also introduce two other forms under the category of basic forms: firstly, the branching tree-like[2] form, or dendrite, which also constitutes a bridge from classical geometry to fractals, a class of non-Euclidean forms often met in nature; and secondly, rhombuses, a class of simple shapes of Euclidean geometry. I acknowledge that the rhombus does not possess a profound visual or logical simplicity in the same way as the square, circle, triangle, and tree-like forms do. The rhombus, however, plays a prominent role in this thesis, especially in the last chapters and appendices. For these reasons, all five aforementioned forms are included in the category of the basic forms in this study.

[1] *Il Saggiatore* "the Assayer", 1623, reprinted in *Discoveries and Opinions of Galileo,* translated with an introduction and notes by Stillman Drake, 1957, pp. 237–238.
[2] With the designation tree-*like*, I emphasize that such a form does not necessarily depict a physical tree, but any entity with a branching and hierarchic structure, either physical or abstract.

Perceptual and Conceptual Forms Representing Nature

Two new concepts introduced in this thesis – *perceptual forms representing nature* and *conceptual forms representing nature* – need to be clarified right in the beginning.[3] The first one refers to mimetic depictions of visible forms and structures that we can see in nature either with our bare eyes or by using instruments. The second one designates the seemingly artificial and typically geometric forms that we humans have invented by accident or constructed with purpose to visualize phenomena or functions of nature in systematic, theoretical, data- oriented, or, broadly speaking, "scientific" ways. I could summarize the essence of these two modes by saying that by using the former, we aim to show how nature *appears*, and by the latter, we aim to show how nature *works*.[4] The perceptual forms representing nature, in other words, are made for the eye, and the conceptual forms the mind.

The Structure of This Study

My own paintings are often connected to these two modes of depiction. On the one hand, I use imitation to represent recognizable objects or scenes in nature. On the other hand, I deploy representations connected to scientific constructions and visualizations. I will speak more of my paintings in Chapter 1. I start with the square, the circle, and the triangle because they constitute the very set most often introduced in the history of visual arts education, especially in teaching the elements of drawing. The triad square-circle-triangle is a visual manifestation of the Platonic concept of eternal and indestructible ideas, and these shapes also

[3] For a relatively long time I referred to these categories as *perceived forms of nature* and *constructed forms of nature*. My principal supervisor, artist Jan Svenungsson, pointed out to me the philosophical weaknesses of such terms: we humans perceive all visible forms, no matter if they are 'constructed' or not. Also: if humans construct some forms, how can they be forms *of* nature? My fellow doctorate students expressed similar types of doubts against such terms. Eventually I reformulated my concepts as they are presented here.

[4] My secondary supervisor, astronomer Tapio Markkanen, pointed out to me that depictions and visualizations are by no means necessary in constructing a functional model of how nature "works". Plain numeric tables of the apparent movements of planets in the sky, for example, were sufficient for ancient Babylonians to build a functional model based on correct observations. They had no need to theorize about or make visualizations of the real movements of planets in three-dimensional space. A similar question arises in quantum mechanics: is there a need, or even a possibility, to form a coherent visualization of nature or "reality" described by this amazingly functional and accurate theory? This non-visual aspect, among other strange aspects, of quantum mechanics was not easily accepted even among some of its original developers. See, for example, Arthur I. Miller, *Imagery in Scientific Thought*, 1986, especially chapter 4: 'Redefining Visualizability', pp. 127–177, and Henk W. de Regt, "Erwin Schrödinger, *Anschaulichkeit*, and Quantum Theory", *Studies in History and Philosophy of Modern Physics*, Vol. 28, No. 4 (1997), pp. 461–481.

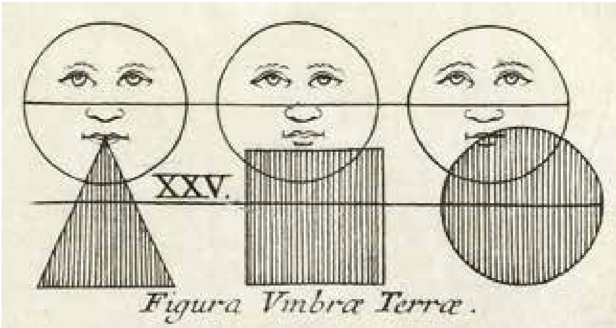


Figure A: Juan Caramuel y Lobkowitz (1606–1682), *Mathesis biceps, vetus et nova*, 1670, Lamina XVII, Figura XXV. The round shape of the Earth can be reasoned from the shape of its shadow (*Figura Umbrae Terrae*) cast on the face of the Moon.

constitute the epitome of modernistic Bauhausian design.[5] In Chapter 2, I discuss the art historical aspects of these three forms.

Simple geometric forms are found not only in the products of human culture, but also in physical nature that is independent of us.[6] In Fig. A, one such form, namely the circle, is seen in the shape of the face of the Moon. As this image illustrates, it is not always easy to draw a line between the perceptual and the conceptual forms representing nature. It is true that during a lunar eclipse, we don't see a triangular or square shadow on the round face of the Moon, but if we see a circular shadow, the round shape of the Earth can indirectly be deduced from that observation, even if the Earth is not directly perceived. It is interesting to note that in this particular 17th century image, for example, it was the triangle and the square, not some other forms, which were used as hypothetical forms of the shadow of the Earth in addition to the correct circular one. More examples of precise geometric forms, either hypothesized or actually met in nature, are presented in Chapters 3 and 4.

I acknowledge that my selection of the basic forms is partly arbitrary since other geometric shapes are also called "basic shapes", for example in visual arts and in the context of many other activities. Such shapes often include, for example, a star, cross, crescent, semicircle, ellipse, parabola, rectangle, parallelogram, chevron, spiral, heart, etc. For the purposes of my main argument, I have nevertheless

[5] The square and the triangle provide forms also for such schemas as four-fold field analysis and the hierarchic pyramid (triangle) of values or needs. In logic, the concept of "circular argument" is well-known. The schema of a tree-like structure is often used to present the logical structure of some "if – then" type of system.

[6] Even the basic forms I have selected don't necessarily belong exclusively to only one of these two categories of representing nature. Consider, for example, the tree-form; a tree-form can represent some physical tree perceived, or it can represent the hierarchic structure of some abstract system.

seen it necessary to limit my study mostly to the five aforementioned forms: the square, circle, triangle, rhombus, and tree-like form. In this thesis, I have neither included, for example, a bent, wrinkled, or otherwise curled line as a form. The short passage about the spiral form in Chapter 4 may be a borderline case in this respect. Through the study of the tree-like form, we face not only the conundrum of the fractals but also that of the "organic forms". I will touch upon the question of organic forms in the end of Chapter 4, but a deeper analysis of the nature of organic forms is beyond the scope of this thesis.

In Chapter 5, I discuss some rather general aspects of forms: how they can emerge as a result of some very simple but universal processes, and how shapes and forms are perceived and evaluated by human perception. In Chapter 6, I give an example of a geometric pattern, the Penrose tiling, which, against all prevailing theories of solid matter, was discovered to exist also in physical nature in a most unexpected manner. A mathematical discovery of mine, which also relates to this pattern, is explained in Chapter 7. I argue this specific part of my research even constitutes a small step in deciphering some lines in the grand book of nature mentioned in the famous passage written by Galileo Galilei (1564–1642) in the very beginning of this Introduction. A general survey of the development of geometric forms and their use in scientific theories is discussed in Chapter 8. I present my Conclusions of this study in the end of this thesis.

Research Questions of This Study

When and how have the circle, the square, and the triangle gained the status of basic forms? What type of existence these forms have in nature independent of the human being? Should they be considered artefacts of human culture or creations of some particular individual minds? Is there, or has there ever been, interaction between the visual arts and sciences with regard to selecting these particular basic forms? On many occasions, new complex forms have replaced older and visually simpler forms in science. What is the role of personal imagination in discovering or in inventing forms and patterns that relate to nature in a "scientific" sense? Is there a correlation between changes in scientific paradigms and changes in the selection of basic forms assumed to describe nature in some particular era? Have we reached the endpoint in the evolution of forms representing nature, or is there still a chance for new forms to emerge in the future?

Basic Forms of Nature

In the realms of perceptual and conceptual forms alike, some forms can be considered more elementary than others in terms of how they represent nature. In some cases, one could even call such forms "basic forms *of* nature". They could be described as geometric manifestations of laws and principles existing in nature, either in visible forms directly accessible to us, or by lying somewhere in the "deeper" or "higher" layers of reality. In some cases, such basic forms have been considered to be material elements reminiscent of Plato's regular polyhedrons, comparable to indestructible atoms, whereas in other cases, their materiality was deemed dubious, a case in point being the idea of celestial spheres.[7]

The term "basic forms *of* nature" highlights, on the one hand, the fact that those who have proposed theories about such forms in relation to nature have sincerely believed that those shapes exist in nature independently of their discoverers and observers. In this sense, "forms of nature" refers to forms belonging to nature. On the other hand, those who do not share this belief tend to say that such forms are rather *invented* and *projected* onto nature by the minds of their inventors and observers. In this sense, "forms of nature" only depict nature.[8] If we can speak of actual physical forms, of the curled horns of a ram, for example, rather than, say, conceptual representations of the periodic table, then I see no problem in speaking about forms of nature in the first sense, as is done, for example, in Chapter 4.

In some respects, the study of these basic shapes can be compared to the study of alchemy and astrology. We can study their histories without believing in them. Nonetheless, it is valuable, perhaps even necessary, to know something of their histories in order to understand how fields like chemistry and astronomy have developed from alchemy and astrology, respectively. In a similar way, we can study how basic forms were used in explaining or in describing nature without believing in those theories. On some occasions, we may even question the existence of such basic forms altogether. However, analogous to the historical connection between alchemy and chemistry, and astrology and astronomy, are historical connections between old theories related to the hypothesized basic forms of nature and many modern and still valid theories of art, perception, and science. Already these historical connections alone justify studying basic forms as a concept and dealing with them in this sense as a category of their own.

[7] C. M. Linton, *From Eudoxus to Einstein: A History of Mathematical Astronomy*, Cambridge University Press, 2004, pp. 21–27.

[8] The theories of the German astronomer Johannes Kepler are an example of the first kind of approach, even if he believed that a supernatural God imposed those forms. I will return to Kepler's theories in Chapter 3. It is hard for me to believe that our contemporary astronomers would either consider, for example, a spherical or ellipsoidal form of a star as anything else than a "form of nature". The German philosopher Immanuel Kant (1742–1804), on the other hand, held the opposite, second kind of view, that "all order and regularity in the appearances, which we entitle *nature*, we ourselves introduce." I will return to Kant's opinion in Chapter 8 and in my Conclusions. In the latter case, we can ask: from where do form, order and regularity come into the minds of those who observe them?

A Figure, a Form, a Shape, a Structure, or a Pattern?

Before going further, I will address some aspects of the terms *figure*, *form*, *shape*, *structure*, and *pattern* with the help of an *ad hoc* example aimed at combining some characteristics of all these terms.[9] The shape in Fig. B may look irregular at first glance, but soon the eye and the mind, quite literally, *figure out* the parts from which this rugged shape is constructed, actually in a very regular manner. This shape is made of squares or "steps" with diminishing sizes (from left to right): each square is one-fourth the area, or half the length of the edge of the previous square. Besides being a figure in the sense of being a numbered illustration in a book, in this case "Fig. B", this image is also a figure in the sense that it is "a shape, which is defined by one or more lines in two dimensions, such as a circle or a triangle."[10]

Fig. B is also a form or a shape in the sense of being "the visible shape or configuration of something (especially apart from colour), the external form, contours, or outline of someone or something, a geometric figure such as a square, triangle, or rectangle."[11] Further, it is easy to imagine the object seen in Fig. B as being "a piece of material, paper, etc., made or cut in a particular form."[12] It is also a structure in the sense of being "the arrangement of and relations between the parts or elements of something complex."[13] The structure of this shape is also a perceivable pattern, as it constitutes "a regular and intelligible form or sequence discernible in the way in which something happens or is done."[14]

This example illustrates how it is not always so straightforward to determine where the dividing line goes between the aforementioned words. In this thesis, I have used the words *shape* and *form* practically interchangeably.[15] The change from

[9] Definitions of these words are found, for example, in the *Oxford Dictionary of English* (*ODE*), a single-volume English dictionary published by Oxford University Press, available online at http://www.oxforddictionaries.com (accessed 2016-07-02). Please note: all online sources mentioned here and later on in this thesis are open access. *The Concise Oxford Dictionary of Current English* (ninth edition 1995) gives slightly different definitions by omitting, for example, the geometric figures such as a square, triangle, or rectangle from the definition of the word "shape". In addition to the terms of figure, form, shape, structure, and pattern, I mention also three other words related to similar concepts. All these words come from Greek. The first is μορφή [morphë], with the meanings "form, shape, appearance, outline, beauty, grace", found in many English compound nouns as the prefix morph-, for example, in *morphology*, "the study of form of things". The second is εἶδος [eîdos], with many meanings, such as "That which is seen: form, image, picture, shape, appearance, look, sight, fashion, sort, kind, species, wares, goods." With its many meanings, this word has produced, and been divided into, many new words, such as idea, ideal, idyll, idol, eidolon (phantom, fancy), and the latest derivate *eidetic*, a word invented in 1920s Germany. The third and last is *schema*, "a representation of a plan or theory in the form of an outline or model", which comes from the Greek word σχῆμα [skhêma]: "form, shape, figure, appearance, fashion, manner, attitude,

Figure B: A shape with a structure in the form of a pattern.

the word shape to structure or pattern becomes necessary only in Chapter 4, where some examples are closer to being structures in nature than forms in nature.

Theory of Forms

In the beginning of this thesis, I called conceptual forms *seemingly* artificial, not *explicitly* artificial. The degree of artificiality of conceptual forms opens up wide philosophical questions and is not as self-evident as it may first appear. Even if we construct such forms, they are still not created *ex nihilo*. If we are seriously conveying information about nature by depicting something which we are not able to perceive visually as a shape, then from where do we acquire this depicted shape? Are the shape and the structure of the depiction mere results of someone's personal taste and preferences, or are they a cumulative product of human culture? Could the form of depiction be somehow "distilled" from nature itself? If we suppose that concepts have their origins somewhere outside the human mind and culture, that is, in "nature", then our point of view comes close to the view presented in *The*

role, character, characteristic property of a thing". The main etymological sources used here and later on in this thesis are Henry George Liddell, Robert Scott *et al.*, *A Greek-English Lexicon* (1843), ninth edition, with a revised supplement 1996, and Robert Beekes, *Etymological Dictionary of Greek*, 2010.

[10] The fourth meaning given to the word "figure" in the *ODE*.

[11] Some combined meanings given to words "figure" and "shape" in the *ODE* and in the *Concise Oxford Dictionary of Current English* (1995 edition).

[12] An example of the usage of the word "shape" given in the *ODE*.

[13] This is the first meaning given to the word "structure" in the *ODE*.

[14] This is the second meaning given to the word "pattern" in the *ODE*.

[15] It is interesting how a Google image search (conducted 2016-07-04) for the phrase "basic shapes" gives very colourful results with a lot of material aimed at children, while an image search for the phrase "basic forms" produces mostly black and white images in the context of drawing and painting, plus many results for basic forms of *something*, for example, basic forms of taekwondo, basic forms of business organizations, etc.

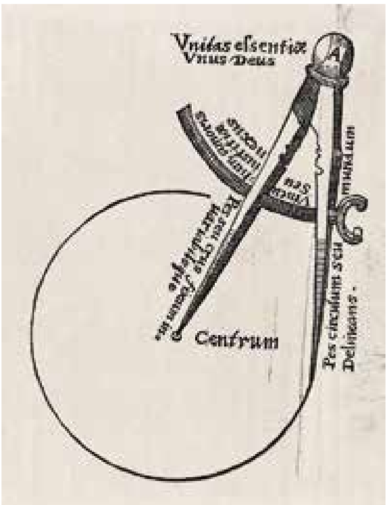


Figure C: Robert Fludd (1574–1637), a divine compass defining a circle, from *Utriusque Cosmi,* 1617, Vol. 2, *De numeris divinis,* p. 28.

Theory of Ideas, or, as it is nowadays more often called, *The Theory of Forms* by Plato (*c.* 428–*c.* 347 BC).[16]

In the Platonic view, reality can be divided into two categories of objects: the ones we connect with our senses and the ones we connect with our intellect.[17] These objects are sometimes called *perceptibles* and *intelligibles*, respectively, the latter ones being equivalents of Platonic *ideas* or *forms*.[18] I connect my concepts “perceptual forms representing nature” and “conceptual forms representing nature” to perceptibles and intelligibles in the following manner: “perceptual forms representing nature” are depictions of nature-related perceptibles, and “conceptual forms representing nature” are depictions of nature-related intelligibles.

In Plato’s dialogues, virtue, justice, good, truth, and beauty are repeatedly mentioned as concepts belonging to the realm of ideas.[19] In addition to these concepts, Plato also mentions as ideas the geometric properties to which we relate through our intellect.[20] The latter category includes, for example, the geometric shapes that we are able to grasp, not only with our intellect, but at least to some

[16] *The Theory of Forms* is presented in the following dialogues by Plato: *Phaedrus* (paragraphs 246–250), *Symposium* (210–211), *Cratylus* (389–390, 439–440), *Meno* (70–87), *Phaedo* (73–80, 109–111), *Republic* (book V 477–480, VI 505–511, VII 514–517 [the cave parable], 522–534 [arithmetic and geometry], X 595–605 [idea, imitation, and art]), *Timaeus* (28–52 [creation]), *Parmenides* (129–135), *Theaetetus* (184–186), *Sophist* (240–241, 246–248, 251–261), *Philebus* (14–18), and in the possibly non-authentic Seventh Letter, that is available online, for example, at http://classics.mit.edu/Plato/seventh_letter.html. Translations of Plato’s dialogues with prodigious analysis, introductions and notes by English scholar Benjamin Jowett (1817–1893) are also available online at (all accessed during mid-July 2015) http://oll.libertyfund.org/titles/plato-the-dialogues-of-plato-in-5-vols-jowett-ed

[17] See, for example, Sir David Ross, *Plato’s Theory of Ideas*, 1961, and Verity Harte, *Plato’s Metaphysics*, in Gail Fine (ed.), *The Oxford Handbook of Plato*, 2008, pp. 191–216.

[18] The English philosopher R. G. Collingwood (1889–1943) used the words “perceptibles” and “intelligibles” in this sense in his posthumously published *The Idea of Nature*, 1965, p. 57.

[19] These concepts are mentioned so frequently in the dialogues that I will not even try to list their locations within the Platonic corpus.

degree, also with our senses. By saying “at least to some degree”, I mean that nobody has ever actually seen, for example, *an absolutely perfect circle* even if we can intellectually grasp what is meant by such an ideal object. An ideal mathematical circle can be defined as the collection of all such points in flat plane, whose distance from a fixed point – the centre – is constant.

With any physical circle, there is at least a theoretical limit to its perfection and accuracy, but with an ideal, mathematically perfect form, there are no limits to its accuracy. With our intellect, we can also grasp “as a completed whole” the idea of the circle, or the idea of the triangle, but we are not able to grasp, for example, the idea of beauty as a completed whole, neither with our intellect nor with our senses. This wholeness does not mean, however, that we are able to understand or even perceive exhaustively all the properties of even such a simple object as a triangle.[21] Nevertheless, we can easily grasp the most essential property of any triangle: being a plain figure with three vertices connected by three lines.[22] In my view, this “completeness” of geometric objects makes a very notable difference between them and some other Platonic ideas.

If we try to give terse definitions to things like virtue, justice, good, or truth, we

[20] Concepts of “roundness”, “figure” and the division of a square, for example, are discussed in *Meno* (paragraphs 74, 82–85) and five regular *Platonic solids* are described in *Timaeus* (paragraphs 51–64).

[21] After more than two thousand years of geometric studies, we still do not know all the “properties” of the triangle. In addition to its three vertices (corners), every triangle contains several other, precisely defined “invisible” special points. There are, for example, the centre of the inscribed circle (the largest possible circle contained inside a triangle) and the centre of the circumscribed circle (a circle which passes through all the vertices of a triangle). The intersection of the medians in turn gives the centre of gravity of a triangle. These three centres may be the most famous, but they are not the only ones. In every possible triangle there are not just these three centres, but also potentially an *infinite* number of new, as yet undefined unique centres just waiting to be discovered. Clark Kimberling, Professor of Mathematics at the University of Evansville, Indiana, USA, keeps an online list of such special points, and his online list identifies over 13,000 currently-known (April 2017) triangle centres at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html (accessed 2017-04-03). And yes, these 13,000 centres are just the “first ones”, and they exist in *every* imaginable triangle, and typically they are not positioned at the same spot.

[22] My emphasis here is completely visual. It is possible to operate with different objects within some other, completely different system and call the object a “triangle” if concepts “point”, “line”, “connect”, etc. are given meanings other than in normal speech. Such “alternative” systems are possible without any logical contradictions. The Finnish mathematician Rolf Nevanlinna (1895–1980) wrote, for example, of differences of visual and conceptual spaces, of the logical structures of geometry, of the interpretation of its basic concepts, and of some non-Euclidean geometries in his books *Space, Time and Relativity*, 1968, pp. 3–48, and *Geometrian perusteet*, [the foundations of geometry], 1973, pp. 3–8.

will probably end up using aphorisms.[23] It is also true that the concept of beauty, for example, is also strongly related to senses and perception, but still we are not able to illustrate "the complete idea" of beauty with an image, as we are able to illustrate "the complete idea" of the circle or the triangle with an image. Let it be also mentioned here that conceptual forms representing nature constitute only a narrow subset within such Platonic intelligibles which are representable with images. My interest in Platonic philosophy in this thesis is therefore limited to those ideas and concepts which are related to nature and which can be represented with images.

This thesis is a study made in the context of artistic research. This implies that questions related to visual arts as well as my own artistic work play an important role in the interdisciplinary setting of the study. In the following chapter, I will highlight the key aspects of my artistic work in relation to the questions revolving around basic forms, nature and nature's representations in either perceptual or conceptual forms.

[23] Using aphorisms for such definitions might after all be exactly the right thing to do, as the very word itself comes from Greek word ἀφορισμός [aforismos] meaning "definition".

1 | A Journey of Diverging and Emerging Paths

In this chapter, I discuss how nature is depicted in my paintings and how I thought my paintings would connect to the written part of this research. A certain divergence grew between my painting and writing, but at the same time, I made a certain discovery in mathematics which has unexpectedly connected to the rest of my research.

My artistic practice is often connected to two modes of depiction. On the one hand, I use imitation to represent directly recognizable objects or scenes. On the other hand, I use representations of more "artificial" scientific visualizations.[24] Sometimes these two categories have fused in my paintings. Hence, my images are not realistic depictions of nature; they are reduced in colour and style. To give an example: I depict trees and plants with their trunks, fruits and leaves as simple beams, ovals and bent flat stripes. As for subject matter, my paintings include recollections from things such as computer games from the 1970s and 1980s, children's toys, signs, graphic design, illustrations, pop art, and hard-edge painting.

Around 2000 I became interested in how the visual language of scientific images representing height, temperature, pressure, density, electric potential or other physical properties with bright false colour[25] schemes often seemed to connect

[24] There is a great deal of artistic freedom in my paintings, meaning that my references and allusions to and modifications of scientific visualizations are done on artistic grounds, not according to their scientific facts or accuracy. To be scientifically valid is not the aim of my paintings, this aim being reserved for my geometric studies, which are presented in Chapter 7 and in Appendices A, B, and C.

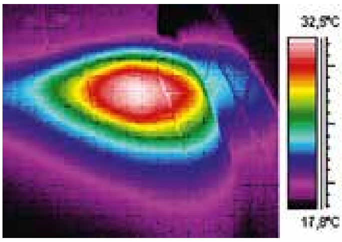
[25] *False colour* means artificial colour coding added to an image during its production to aid interpretation of the subject. Colours are "false" because they rarely have any resemblance to colours possibly visible in the subject. Many maps as well as satellite and thermal camera images are typically rendered with false, or pseudo-colours. See, for example, Michael Marten *et al., Worlds Within Worlds; A Journey into the Unknown*, 1977, p. 201; 'Beyond Light – False Colour', or see https://en.wikipedia.org/wiki/False_color (accessed 2017-03-06).

Figure 1.1 Markus Rissanen, *Blueprints for Landscapes*, [M125], 140x160cm, acrylic on canvas, 2008.

with the images of the aforementioned fields of visual culture. This unexpected purely visual connection between such different realms opened up a new, colourful and fruitful world to me. I was able to play with ideas and objects, which at face value had nothing to do with each other but nevertheless shared a visually coherent universe ripe with playful associations. One example of such an approach is the hard-edge executed painting *Thermodynamic Simulation of the Pastilli Chair* (2000), which combines a 1960s plastic chair designed by Finnish Eero Aarnio (b. 1932) with an imaginary thermodynamic simulation. See Fig. 1.2 (right).

Quite often I have used circles (or ellipses) to mark the ground in my paintings. I have used such a solution to represent perspective without the cliché-ridden use of straight lines converging at the horizon. At the same time it has been possible also to combine a sense of aerial perspective with an effect where the subject matter, sometimes almost literally, seems to be in the "limelight". See Figs. 1.1 and 1.2 (right), both of which contain a luminous, circular "phenomenon" visible on the ground, radiating away from the apparent "centre" of the ground. Artistic freedom even allows for the reversal of such a luminous effect without any apparent problem; in Fig. 1.2 (right), for example, the luminosity of the ground actually *increases* away from the "centre". This increase leaves a dark shadow below the object in the foreground and produces an illusion of an aerial perspective near the horizon.

Quite often I have also used another oval shape to represent objects of nature in extremely reduced form. The boundary of the oval in question is almost always constructed from two halves of a circle connected with straight lines.[26] The relation of height to width can vary, but most often it is 1:2 or 2:3. The nature of this oval form differs from that of a circle or an ellipse, as it has straight lines which an ellipse or a circle do not have.[27]



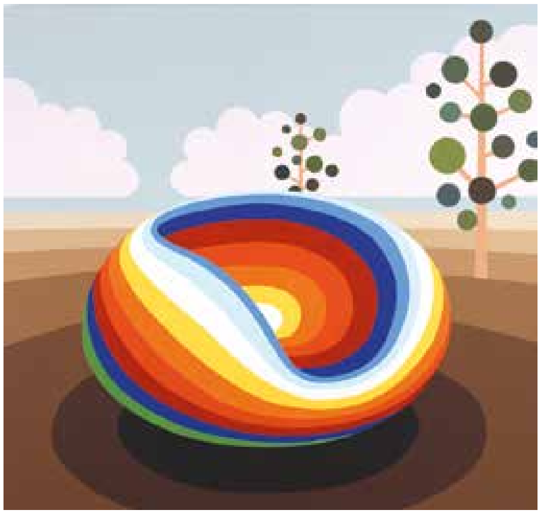


Figure 1.2 A false colour image representing temperatures of a floor with a hot water pipe leaking inside it (left), and Markus Rissanen, *Thermodynamic Simulation of the Pastilli Chair*, [M12], 122x130cm, acrylic on canvas, 2000 (right).

I also consider it to be more dynamic than a circle because the form of the circle doesn't have any direction or specific orientation. On the other hand, I experience a precise, mathematically correct ellipse to be visually somehow *too* dynamic, restless, or even ephemeral compared to the oval I have been using. My oval always carries along a recollection of its cultural origin. It is a shape that cannot be found in nature; it is not a natural form. It is a culturally constructed form, an artefact. At the same time, it is calm and stable, yet more versatile and dynamic in use than a circle. Some people have interpreted it as a "pill". Even if I have consciously used bright colours and sometimes even direct references to psychedelics, my intention has never been to refer to drugs. Such subject matter has never interested me as such, or perhaps some bright visual effects connected to them, at the most.

[26] My supervisor Jan Svenungsson brought to my attention *blp*, or "blip" used by the American artist Richard Artschwager (1923–2013) from 1967 onwards. I was not aware of it before 2016, and there are similarities between its appearance and the shape I have been using. Artschwager's earlier blips seem to have had a slightly blunt ends; see, for example, photographs available at http://momaps1.org/exhibitions/view/165 but later on they seem to have developed more (perhaps even completely) circular ends, albeit with a longer body this time; see, for example, the documentary video at http://whitney.org/Exhibitions/RichardArtschwager/Blps (both accessed 2017-04-03). Nonetheless, the greatest difference between the blip and my shape is that the former were presented as abstract marks (or undefined symbols) in urban environments, only one at a time, whereas I have used the latter as a unit of some unspecified visual data, in groups and in a visually uniform environment.

[27] Whereas an ellipse is a mathematically defined precise curve, an oval refers to anything having a rounded and slightly elongated shape without any specific parameters.

Figure 1.3 Markus Rissanen, *The Basics of Quantum Biology*, [M50], 140x160cm, acrylic on canvas, 2003 (left), and *Flow*, [M77], 70x80cm, acrylic on canvas, 2004 (right).

Pseudo-texts

Besides representing objects from nature, the oval shape also has another important function in my works. The same oval also serves as a kind of "basic module" in building up something one might describe as "pseudo-text". In this use, the oval has often elongated to a straight line with round ends. These lines form together something that *resembles* lines of text seen from some distance; see Fig. 1.3 above.[28] This use of rounded-end lines of my pseudo-text has many associations for me. A lot of information in our western culture has been presented as dots and lines of varying length and size. Good examples are Morse code, punched cardboard cards used to program the first mechanical looms, that is, jacquards, and later, electronic computers. More recently the information in CDs and DVDs has been stored using such binary patterns; see Chapter 5, Fig. 5.6.

Technical development and optical readers have lately produced a great variety of new visual coding technologies to share and access information online. These technologies include, for example, QR-codes, among others. I have intentionally refrained from using references to such newer agents of information, as reading them requires additional *equipment*. In my paintings, the viewer can see, with unaided eyes, that the "pseudo-texts" have no readable content.[29] Pseudo-texts in my works present only the *form* of information but omit the content. The circular

[28] The Swedish designer Ola Wihlborg, for example, has also used a similar pattern in his futuristic Code Basket. See, for example, http://www.asplundstore.se/products/code-basket (accessed 2018-18-18).

ends of such shapes are also a reference to the 1970s (to the middle 1980s) habit of using such rounded forms in many areas of design.

At some point, the connection between scientific visualizations and images from the other fields of culture started to interest me from another point of view as well. It seemed only logical that the visual characteristics of such things as children's toys, signs, traffic signs, picture symbols, or graphic designs presenting information in general were often similar to those of some scientific visualization. Many of these frequently use simplified forms and colours to avoid distraction, to focus attention, and to convey their message as clearly as possible. [30] I pondered, for example, why the iconic metro-map of London designed in 1931 by the technical draughtsman Harry Beck (1902–1974) functioned so well while being totally matter-of-factly yet at the same time joyful, even "childish" in its appearance.[31] A detailed view of this diagram demonstrates the similarity between the "pseudo-information" depicted in my paintings and the style used by Beck in his "info-graphics", compare Figs. 1.3, 1.4, and 1.5.

In addition to thinking about these scientific visualizations created from circa 1950s onwards with computers, cathode ray tubes and other kinds of "modern"

Figure 1.4 Markus Rissanen, *Cooled Particles*, [M28], 37x41cm, acrylic and epoxy resin on MDF, 2002 (left), and *Branch of Natural Logic*, [M85], 100x110cm, acrylic on canvas, 2004 (right).

[29] From a hypothetical "pseudo-QR-code", if done well enough, a typical non-expert viewer cannot deduce without properly functioning equipment whether one is seeing a real functional code or just a non-functional look-alike.

[30] In this context, the "message" of children's toys could be something like: "I'm nice and interesting – play with me!"

Figure 1.5 A detail of the 1949 version of the map of the London Underground. Harry Beck, creator of the diagram, considered this perhaps the best of all of his versions.

technologies, I evolved questions about other, even simpler and perhaps even more profound types of visualizations. I began to search the history of scientific visualizations and asked myself how invisible phenomena are given a visible form in the first place. What kinds of forms do we tend to give to something we are able to detect or observe, but are not able to mimetically depict? What is the role of geometry and its elementary shapes in these depictions?

These questions led me somewhere around 2005 to develop the conceptual distinction between the two modes of depicting nature introduced in the beginning of the thesis, namely *perceptual forms representing nature* and *conceptual forms representing nature*. I was interested in how we give visible form to some observable phenomena which nevertheless cannot be depicted mimetically.

[31] See Ken Garland, *Mr Beck's Underground Map*, 2003, and Mark Ovenden, *Metro Maps of the World*, 2003. Similar distortions and simplifications of the geometric space were already used in the ancient Roman road maps called *itineraria*. One fine example of such a map is the *Tabula Peutingeriana*, preserved at the Österreichische Nationalbibliothek, Vienna. The success and visual simplicity of Beck's original map, or more precisely, diagram, have produced innumerable imitations, allusions, and parodies.

Artistic Research in this Study

When I started my studies for the doctorate in 2007 I believed that my artistic work and thesis would have a more direct connection and that they would evolve in greater unison, but this did not happen. In retrospect, I don't find it appropriate to call my study "practise-based" or "practice-led", as artist-researchers' studies often are.[32] The Finnish designer and Doctor of Arts Turkka Keinonen presented eight possible models through which art and research may interact.[33] Using Keinonen's terms, I describe the relationship between my art and my research as having a "common denominator": my interest in geometrical issues and the cultural history of forms. I see them as "overlapping fields".[34]

The main themes of this study emerged from my artistic practise, but the written and thus "fixed" research plan somehow stiffened my artistic work. Perhaps I felt that my paintings should flow seamlessly in line with the research plan and later on with the thesis. My paintings were in danger of becoming illustrations of pre-determined themes. In the beginning of my doctoral studies, I became interested in depicting forms of certain, often science-related, visualizations of information without providing the supposed "content" or "data" in the painting. One such instance was the *Genotype* series painted in 2006–2007, one example of which is seen in Fig. 1.6.

Figure 1.6 Markus Rissanen, *Genotype*, [M120], 45x40cm, acrylic on canvas, 2007.

With this series, I wasn't trying to exclaim anything about genetics or genotypes as such; I was simply interested in how genes located in chromosomes were depicted in some visualizations as stripes coded either with simple colours or with letters and numbers.[35] Such an approach had its difficulties, as the "serious" title *Genotype* can easily suggest that there is some solid

[32] For a discussion about the small differences between these terms, see, for example, Maarit Mäkelä and Sara Routarinne (eds.), *The Art of Research*, 2006, p. 12 –15.

[33] The models presented were 1) research interpreting art, 2) art interpreting research, 3) art placed in a research context, 4) research placed in an art context, 5) art contributing to research, 6) research contributing to art, 7) the common denominator, and 8) overlapping fields; see Turkka Keinonen, "Fields and Acts of Art and Research" in Mäkelä and Routarinne (2006), p. 45–54.

[34] *Ibid.*, p. 51–53.

[35] I found such illustrations, for example, in the following 1970s *Scientific American* issues: Dec. 1976, pp. 102–113, Dec. 1977, pp. 54–67, Nov. 1978, pp. 52–59, and in numerous examples in the October 1985 issue with the specific theme "The Molecules of Life".

information provided, unlike in some other works with more "fanciful" titles, such as *The Basics of Quantum Biology*, which was to refer to a totally fictive field of science, the name of which I invented *extempore* at the time.[36]

For some reason, I have never been very interested in using the basic geometric forms in my paintings, and I felt this as a lack during my research. My person and interests constitute one common denominator for my art and theory. There are certainly also some overlapping fields in my art and theory as they both relate to, for example, scientific visualizations and depictions of nature. Nonetheless, the basic forms as such were missing from my artistic production. The relationship between art and theory seemed to become even more complicated as a third area of research developed within my studies. In addition to the artistic work and the written part, there evolved quite independently a mathematical element with a strong visual character. This mathematical, or geometric, element was inherently connected to some essential properties of the simple shapes of Euclidean geometry, such as the square, circle, and triangle. Hence, in the end, this development unexpectedly clarified the situation and provided another overlapping field, or bridge, between art and theory.

Geometric Research in this Study

There is a simple and well-understood concept in mathematics called *periodicity*. In mathematics, all patterns that repeat in a linear way only, are said to be periodic. Patterns can repeat periodically in one, two, or three dimensions.[37] Many patterns consist of two-dimensional closed forms only. If the number of different forms used is finite, the whole configuration is called *tiling* after its real-life counterpart.[38] A tiling is typically periodic, but not necessarily. Some elementary tilings are obtained

[36] Later I found out that there actually already exists, or at least has been proposed to exist, such a field of research as "Quantum Biology"; see, for example, Neill Lambert *et al*, "Quantum Biology", *Nature Physics*, Vol. 9, Issue 1 (2013), pp. 10–18, also available online at http://www.nature.com/nphys/journal/v9/n1/index.html (accessed 2016-08-22). Around 2001–2002 I read the book *Quantum Evolution*, published in 2000, by Johnjoe McFadden, and, if I remember correctly, this book inspired me to formulate my own "fanciful" title in 2003. During my research I found out that in 2014 Johnjoe Mc Fadden, co-authored with Jim Al-Khalili, even wrote a whole book with the title *Life on the Edge: The Coming Age of Quantum Biology*.

[37] I will limit my examples to three dimensions in this thesis. In more advanced mathematics, there is no limit to the number of possible dimensions; their number can even be infinite.

[38] See, for example, Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, 1987. This 700-page book has been an authoritative opus in its field already for three decades, having been also out of print for a long time until Dover Publications issued a second edition in June 2016.

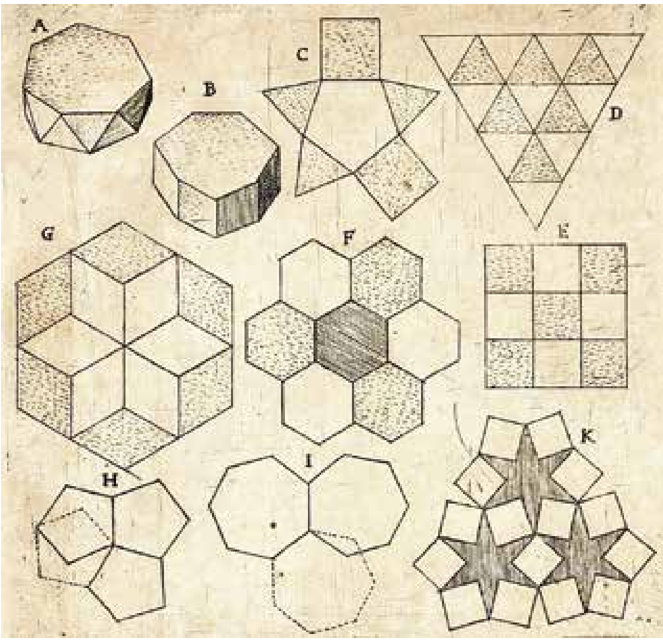


Figure 1.7 Johannes Kepler, tilings and polyhedrons from *Harmonices Mundi,* 1619.

by using elementary shapes such as the triangle, square, hexagon, and rhombus. As many such shapes and patterns can be observed in physical nature, we can use them as perceptual forms representing nature. We may also use such shapes as conceptual forms representing nature. But in addition to these two modes of depiction, these geometric shapes and patterns can be used to visualize some facts of Euclidean geometry. As the physical world is, by all practical measures, at least as far as we perceive it, Euclidean, such geometry with its objects illustrates some "boundary conditions" upon which nature is based.[39]

Johannes Kepler (1571–1630), a German astronomer with a strong appetite for geometric regularities, also depicted tilings in his *Harmonices Mundi* in 1619.[40] Kepler's book contains a suitable illustration (Fig. 1.7 above) of the following facts concerning tilings. In the following, we take no notice of the different shadings used in these tilings. One fact of Euclidean geometry is that six equilateral triangles can be positioned, edge-to-edge, leaving no gaps, around a shared vertex, or corner-point: see Fig. 1.7 (D). With other regular polygons, four squares can be

[39] I am talking of the physical world, as it is perceived in an every-day scale. I am not saying that the geometry of the physical world is Euclidean in all scales; most probably it is not.

[40] Scanned versions of the original *Harmonices Mundi* are available online, for example, at https://archive.org/details/ioanniskepplerih00kepl and another, persistent link at http://dx.doi.org/10.3931/e-rara-8723 (both accessed 2015-09-01).

positioned around a shared vertex (E), and with regular hexagons, only three can be placed tightly around a point (F). In each of these cases the pattern, or tiling, can be continued to cover the whole plane uniformly. It is not possible to make a tiling using only regular pentagons (H), heptagons (I), octagons, and so forth, as there will be either some overlapping or gaps left in between such polygons (H and I), and overlapping or gaps are not allowed in a mathematical tiling. There are some important properties involved in all periodic tilings, and especially in the case of equilateral triangles, squares, regular hexagons, and rhombuses, the concept of *rotational symmetry* enters the picture. The tilings in Fig. 1.7, made of the triangle, square, and hexagon, have sixfold, fourfold, and threefold rotational symmetries, respectively. Note that in this case the centre of the rotation is fixed in the vertex, where the neighbouring polygons meet, *not* in the centre of a polygon.

It was a well-established theoretical and observed fact that all atoms in a crystalline solid matter are organized in periodically repeating units.[41] Like rhombuses, regular hexagons, squares, and equilateral triangles, a perfect crystal can also have only a two-, three-, four-, or sixfold rotational symmetry, respectively. According to the classic crystallography, a repeating five-, seven-, eight-, nine-, or larger *n*-fold rotationally symmetric atomic structure is not possible in solid matter.[42] The British mathematical physicist Sir Roger Penrose (b. 1931) discovered in 1974 a nonperiodic tiling which possessed many properties of periodic tilings made of regular triangles, squares, or hexagons yet having a fivefold rotational symmetry.[43] A Penrose tiling can be made, for example, by using two specific rhombuses. Ten years later, in 1984 the Israeli chemist Dan Shechtman (b. 1941) published a remarkable paper about the discovery of the first known solid matter which seemed to possess precisely the same bizarre fivefold symmetric structure as the Penrose tiling.[44] Such nonperiodic solid matter was soon named "quasicrystals", as their structure was not periodic but *quasiperiodic* instead.[45] Later, even more quasicrystalline materials were found with other non-crystallographic rotational symmetries, such as eightfold, tenfold, and twelvefold rotational symmetries. In

[41] The contents of this and the following page are explained in more detail in Chapter 6.

[42] See, for example, John G. Burke: *Origins of the Science of Crystals*, 1966, p. 3, or P. P. Ewald (ed.), *Fifty Years of X-ray Diffraction*, International Union of Crystallography, 1962, p. 21, which is also available online at http://www.iucr.org/publ/50yearsofxraydiffraction (accessed 2015-10-06).

[43] Martin Gardner, "Extraordinary non-periodic tiling that enriches the theory of tiles", *Scientific American* 236 (January 1977), pp. 110–121.

[44] D. Shechtman, I. Blech, D. Gratias, J. W. Cahn, "Metallic Phase with Long-Range Orientational Order and No Translational Symmetry", *Physical Review Letters,* Vol. 53, No. 20 (Nov. 12, 1984), pp. 1951–1953.

[45] These concepts are also explained in more detail in Chapter 6.

2011 Dan Shechtman received the Nobel Prize in chemistry for the discovery of quasicrystals.[46]

As a hobby, for two decades I had already been studying some geometric properties of tilings with "strange" rotational symmetries, but nothing serious had come out of these studies. After I heard of the Shechtman's Nobel Prize, however, my interest was reawakened. I realized that quasiperiodic tilings and quasicrystals fall nicely within the sphere of my doctoral studies as they directly relate to simple geometric forms and structures found in nature. To my knowledge, quasiperiodic tilings with non-crystallographic rotational symmetries had already been found for values n = 5, 7, 8, 9, 10, and 11, but a general solution for all n was not known.

In 2012 I discovered a way to construct a quasiperiodic tiling with an arbitrarily large n-fold rotational symmetry.[47] Later, I made contact with professional mathematician Jarkko Kari, a Professor of Mathematics from the University of Turku, Finland, who managed to prove my discovery – in a strict mathematical sense. The discovery and its proof were published in our co-authored paper in the peer-reviewed *Discrete & Computational Geometry*, Vol. 55, Issue 4, June 2016, pp. 972–996.[48] Due to its rather technical nature, the paper is included in the thesis only as an appendix. The discovery and its background are explained in more accessible terms in Chapters 6 and 7.

Thus, my artistic research partly turned into mathematical research, and the basic forms that had been quite absent in my paintings emerged in my geometric studies. In the end, my tiling system even contained the very forms which I consider as basic forms in this thesis. As the detailed presentation of my tiling system in Chapter 7 shows, the system consists of rhombuses. The rhombus, in turn, can be seen as a fusion of two identical isosceles triangles. Furthermore, my system contains rhombuses with different vertex angles. In some cases, the angle is right, highlighting the fact that the square is also a rhombus. The rhombus can, in other words, be seen as a "slanted square".[49] The rotationally symmetric nature of my tilings causes them to have an overall circular shape. Even the tree-like pattern implicitly exists in the logical structure of my tilings: every substructure contains copies of the whole form, as the tree-form also does. Before going further in describing my geometric system, I will examine the basic forms from other perspectives, starting with an art historical survey.

[46] http://www.nobelprize.org/nobel_prizes/chemistry/laureates/2011/press.html (2015-12-01)

[47] In Chapter 7, I will explain how this discovery unfolded.

[48] The paper is available at http://link.springer.com/article/10.1007/s00454-016-9779-1 (behind a pay-wall), a free pre-review version of the article is also available online at http://arxiv.org/abs/1512.01402 (both accessed 2017-03-27).

[49] In Finnish, the rhombus is *vinoneliö*, literally "slanted square".

2 | Basic Shapes in the Visual Arts

In this chapter, I discuss how simple geometric shapes have been used as visual tools in the visual arts for several millennia. I will use the analysis of the proportions of the human body as an example of such usage. The theory of the assumed correspondence between the basic forms and colours by Wassily Kandinsky is also presented. I start this chapter by analysing the famous quote by Paul Cézanne, which has had a great impact upon some interpretations concerning the geometric solids and modern painting. I will demonstrate that there is a persistent misconception concerning the cube in many such interpretations.

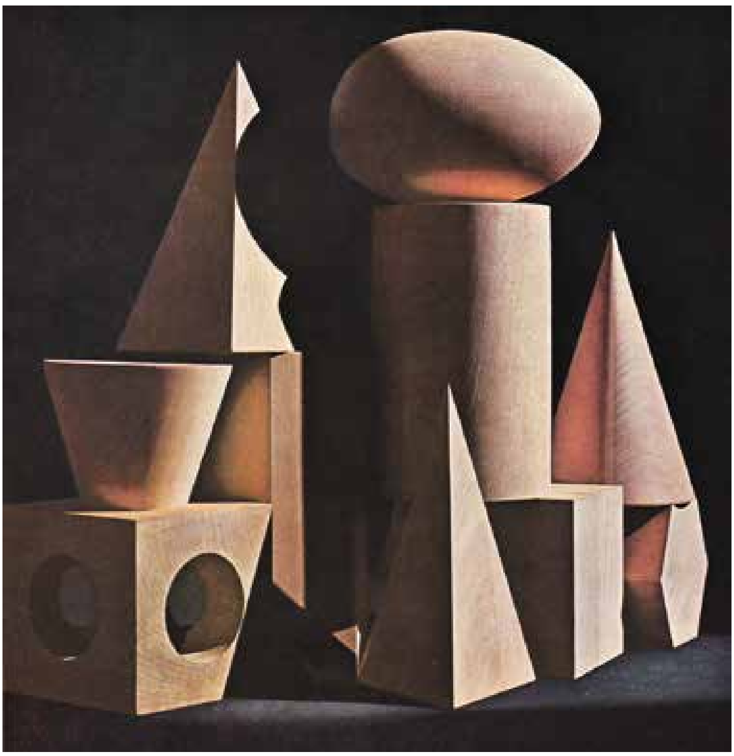


Fig. 2.1 An exercise in Mathematics or a composition in the Fine Arts?

Figure 2.2 Godfrey Sykes and J. Emms, *Model Drawing* (1863), a lunette painting in the South Kensington Museum, nowadays the Victoria and Albert Museum.

Three Round Solids

In his most famous and much-much-quoted passage, the French painter Paul Cézanne (1839–1906) advised a certain younger colleague in the following way:

> *May I repeat what I told you here: treat nature by means of the cylinder, the sphere, the cone, everything brought into proper perspective so that each side of an object or a plane is directed towards a central point. Lines parallel to the horizon give breadth, whether it is a section of nature or, if you prefer, of the show which the* Pater Omnipotens Aeterne Deus *spreads out before our eyes. Lines perpendicular to this horizon give depth. But nature for us men is more depth than surface, whence the need to introduce our light vibrations, represented by the reds and yellows, a sufficient amount of blueness to give the feel of air.*[50]

Cézanne was not very happy in providing his answers to the theoretical questions concerning the art of painting, which his younger admirer and colleague Emile

[50] Paul Cézanne, *Letters,* edited by John Rewald, 1995, pp. 300–301. The letter in question is dated in Aix-en-Provence April 15th, 1904.

Bernard (1868–1941) eagerly posed to him.[51] American art historian Theodore Reff (b. 1930) wrote of this famous passage and its later interpretations in his 1977 article "Cézanne on Solids and Spaces".[52] In his article, Reff demonstrates how Cézanne was mainly repeating what he had himself learned in typical European art education. For example, French artists Pierre-Henri de Valenciennes (1750–1819) and Jean-Philippe Voiart (1757–*c.* 1840) recommended that students start by learning first to draw three-dimensional basic shapes: cubes, cylinders, and spheres.[53]

Cézanne speaks of positioning objects in relation to the horizon and how to give the space depicted breadth and depth. Reff mentions that one traditional treatise of perspective was Jean-Pierre Thénot's (1803–1857) *Principes de perspective pratique* (1832), a book Cézanne is reported to have owned.[54] After representing elementary geometric objects such as the point, angles, lines, surfaces, polygons, the circle, and simple solids, Thénot begins his lecture on perspective: "Perspective is the art of representing on a surface called a picture the outlines of object as they appear to us. The first thing to be done is to decide the horizon."[55]

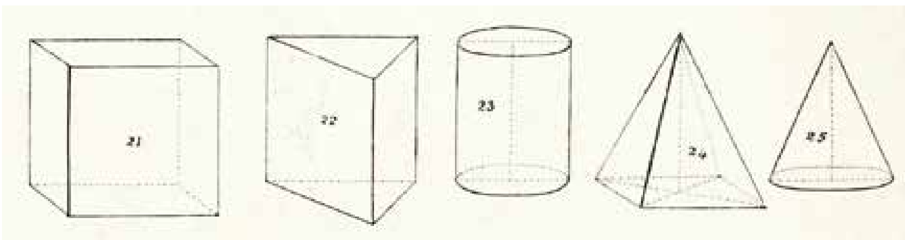


Figure 2.3 Jean-Pierre Thénot, geometric solids depicted in Plate 1 of his book *Practical Perspective for the Use of Students* (1834).

[51] *Ibid.*, pp. 300-301; footnote by editor J. Rewald: "Always deeply involved in philosophical and religious thought, Bernard seems to have had long theoretical discussions with Cézanne, which he attempted to continue in his letters. Although Cézanne had little taste for such speculations and discreetly made this apparent in his answers, Bernard's questions did in fact make him express his own views about painting. Bernard published his 'Souvenirs sur Paul Cézanne', but did not realize how critical Cézanne was about his young admirer's work."

[52] Theodore Reff, "Cézanne on Solids and Spaces", *Artforum*, October 1977, pp. 34–37.

[53] Reff (1977), p. 35. Sometimes referred also as *Jacques*-Philippe Voiart, Voïart, or Voyart.

[54] *Ibid.*, p. 36. Thénot's book is available online in the 1834 English and the 1845 second French edition; see https://archive.org/details/bub_gb_BLdjy1R2MZIC (French edition) and https://archive.org/details/practicalperspec00th (English edition), both accessed 2016-07-29.

[55] Thénot, *Practical Perspective for the Use of Students*, 1834, pp. 10–11.

One Solid Mistake

There is something in Cézanne's quote which has gone unnoticed by numerous people during the last one hundred years. Cézanne spoke of *the cylinder, the sphere*, and *the cone*, but nowhere in his letter does he mention *the cube*. It is ironic that the radical art movement, which held Cézanne in highest esteem, got its very name from this particular "missing" solid.[56]

Theodore Reff estimated that this misconception had already started to develop around 1920.[57] That the error was already well cemented a few years later can be inferred from the text of the French cubist painter Albert Gleizes (1881–1953), published in 1927, where he states: "Cézanne spoke of the cylinder, cube, and sphere, thinking that their purity could unify everything."[58] But most probably Cézanne was not referring to eternal Platonic solids and not even to the old practise of reducing the shapes of the objects into simplified geometric shapes in order to see and draw their essential shape better. Even in the latter case, the idea of using simple geometric shapes as a tool to organize visual impressions would have been nothing new: simple geometric shapes had been used in teaching arts and especially drawing, at least in Europe, for about two centuries before Cézanne.[59] The idea of using physical geometric solids in the teaching of drawing seems to have its origins in 1830s Paris.[60] Of course the use and study of geometric forms

[56] John Golding, *Cubism: A History and an Analysis, 1907–1914*, 1988

[57] Reff (1977), p. 35.

[58] *Ibid.*, p. 35. It is obvious that the erroneous version had long ago attained a flourishing life of its own. The difference went unnoticed even from such a famous art critic and philosopher as Arthur C. Danto (1924-2013), who wrote in his essay *Pictorial Representation and Works of Art*: "[W]e have the famous conversations with Emile Bernard, in which Cézanne speaks of nature as so many cubes, cones, and spheres—a kind of Pythagorian [sic] vision of the ultimate forms of reality, never mind what the senses say and conventional paintings show. Not many years after these geometrical speculations, the Cubists were painting the world in pretty much those terms." Danto's essay can be found in Calvin F. Nodine and Dennis F. Fisher (eds.), *Perception and Pictorial Representation*, 1979, p.11.

[59] I will discuss this subject later in this Chapter.

[60] The British politician and educationist Sir James Phillips Kay-Shuttleworth (1804–1877) delivered a lecture "The Constructive Method of Teaching" at Exeter Hall, London, April 19th, 1842, and it was published in *The Saturday Magazine*, (London), No. 647, Supplement, July 1842, pp. 41–48. In his lecture, Kay-Shuttleworth explained how the inspector of elementary drawing schools of Paris, M. Dupuis had observed how students who were very skilled at making beautiful drawings after flat models were, nevertheless, very poor in drawing from three-dimensional natural objects. Kay-Shuttleworth: "M. Dupuis devised a series of models, the intention of which was to practice the pupils of the drawing schools of Paris in drawing form, from the models so placed as to enable the master, not by reference to geometry, but simply by reference to certain general laws of light, –some obvious common-sense views of the subject– to teach the general laws of perspective." Monsieur Dupuis seems also to have taken active measures to commercialize his invention. *The London and Paris Observer* (Paris)

and solids in architecture, decoration and in other visual arts has a much longer history (and prehistory) as such, if the pedagogic aspect is not the main point of view; just think of Stonehenge and the Pyramids of Egypt, for example.

It seems most likely to me that Cézanne was just explaining how to make the illusion of *roundness* to give objects in the image a sense of *volume*. The cylinder, the sphere, and the cone all give the sensation of smoothly curving surfaces seen from different angles – but not the cube, which looks angular from all angles. As Reff wrote: "For it is clear that, at least in his letter of April 15th, Cézanne chose the cylinder, the sphere, and the cone not as geometric solids of Platonic purity, but as forms whose curving surfaces reduce continually from the eye."[61] I interpret the fact that so many authors later specifically added the cube – a natural derivate of the square – on Cézanne's plate as a good example of the archetypical power of this form. But even the archetypes have their origins.

A Form is Born

Like Pallas Athena, who was born mature and fully armed from the forehead of Zeus, so the triangle, square, and circle seem to emerge into existence as completely developed figures. There seems to be no trace of evolution left in them from any apparently simpler two-dimensional, regular or irregular, shape or shapes, which could have evolved into these three pure forms. Among all forms, whose potential number is unquestionably infinite, some forms always look more fundamental than others. This fact has often been used in art theories, art pedagogies and in drawing manuals.

Simple geometric forms are often called basic shapes, at least in contemporary fields of visual arts like architecture, drawing, painting, and design. In these fields, the triangle, square, and circle are usually considered as *the* basic shapes or forms.[62] However, the expression "basic shape" is not unambiguous or universally used. For example, in the field of perception psychology, even if the field itself has a long

announced in its No. 480 issue, July 27th, 1834, p. 480, of new patents granted by the French government in 1833, among which was also one granted for five years to "J. M. Dupuis, of Paris, for a new method of drawing from busts." Exact location: p. 480, right column, 11th row from the top. Both magazines, *The Saturday Magazine* (1842) and *The London and Paris Observer* (1834), were available online via Google (both read 2015-08-26), but their URLs seemed very messy and ephemeral.

[61] Reff (1977), p. 35.

[62] In Finnish, these forms are sometimes also called *valiomuodot*, with meanings such as "pre-eminent forms", "forms of excellency", or "elite forms". Tapio Markkanen remarked that it is very likely that the Finnish philosopher Eino Kaila (1890–1958) specifically invented this word. Eino Kaila founded the psychological laboratory at the University of Helsinki and is sometimes considered to be the founder of modern Finnish psychology.

tradition of studying the perception of such simple forms, the expression "basic shapes" seems not to be used at all.[63]

When we talk of basic forms, or any two-dimensional forms, are we actually talking about the thin line which defines the limits of such a form, or are we talking about the area which is surrounded by this thin line? In material reality, we can compare this to the situations where a triangle is either folded from a wire, cut from a thick paper or modelled from, let us say, clay. Cutting the form out of paper removes the "unnecessary" material, leaving the "interior" of the form. Modelling clay organizes the mass into a new spatial form, but folding wire into a triangle creates an illusion of the triangular "interior". Naturally, we have to admit that all of them legitimately define a form. In the following, I shall stay in two-dimensional forms and not refer to three-dimensional solids when speaking of "forms".

One way of drawing a form on paper with a pen is just to start colouring (filling) the paper with the pen. Eventually a two-dimensional coloured area may be defined, which can be seen as a form. More often this colouring operation is done only after another manoeuvre with the pen. I mean here of course a thin line drawing in which a pen touching a paper marks a point, which by continuous movement of the pen grows into a line, and this line can enclose the two-dimensional area, the result of which is seen as a form. In some respects, this other operation resembles the "wire" model. I claim that it is not easy – perhaps not even possible – to say which one is seen as the form: the thin outline or the area enclosed? Imagine, for example, a silhouette: *which* is seen as the form – the fine outline, or the black "interior"?[64] I believe the answer is both and neither: there is no outline without an enclosed area, and there is no enclosed area without an outline, even when the outline is *illusory*, like the perimeter we see enclosing the white rectangle in Fig. 2.4 (above). However, in theories or pedagogies of form in fine arts this "point-line-plane" approach is more often seen than the "paper-cutting" or the "clay-forming" approach.

[63] Based on my personal observation concerning terminology and keywords in the publications of perception psychology: the expressions *basic shapes*, or *elementary forms*, are totally absent. A personal communication with Ilmari Kurki, a university lecturer of perception psychology in the University of Helsinki, further reinforced this belief (2014-04-24).

[64] I suspect that there is no single "scientific" opinion to settle such a question. Furthermore, I suspect that the very division of "outline" and "area" is already an oversimplification of some sort, if seen from the perspective of perception psychology.

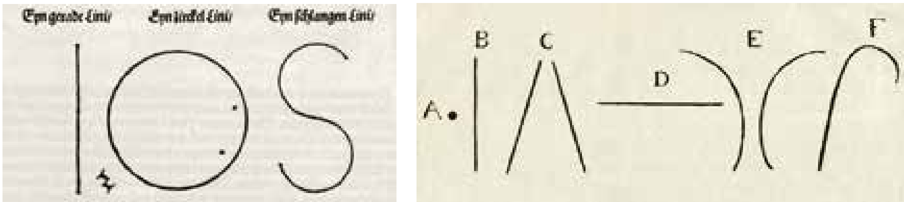


Figure 2.5 Lines depicted in Albrecht Dürer's *Underweysung der Messung* 1525 (left), and lines depicted in Gerard de Lairesse's *Grondlegginge Ter Teekenkonst* 1701 (right).

The Point, the Line, and the Plane

It is most often via zero-dimensional points and one-dimensional lines that the two-dimensional "basic forms" are introduced in many manuals of art.[65] The example *par excellence* is of course Wassily Kandinsky (1866–1944) and his book *Point and Line to Plane*.[66] The reader may now feel that the forms constructed in this point-line-plane manner are interpreted rather as fundamental objects of all art and not just as visual shapes used in drawing and painting. However, this point-line-plane approach was by no means an invention of modern artists and theorists. In this geometric point-line-plane respect, modernists can be regarded as continuing a long and well-established tradition not only within western art but also within western science, starting at least from Euclid (active *c*. 300 BC). Albrecht Dürer (1471–1528), after praising Euclid, started his geometric treatise *Underweysung der Messung* (1525), known in English simply as *The Painter's Manual* by introducing a single point and a straight line.[67]

His second image depicted three different lines looking very much like letters I, O, S, from some 20th century modernist typography, marked as "Eyn gerade Lini / Eyn zirkel Lini / Eyn schlangen Lini", meaning "a straight line / a circular line / a snake line", see Fig. 2.5 (left). A similar geometric point-line-plane approach to forms is also found in an influential treatise on drawing, *Grondlegginge Ter Teekenkonst* (1701), "The Principles of Drawing", written by Dutch painter

[65] It is interesting to think about how the other possible approach of defining a form in a flat two-dimensional plane, namely the silhouette, or the freer "cut-and-paste" method, could be applied in pedagogical situations.

[66] The original *Punkt und Linie zu Fläche* was published in German by Bauhaus in 1926. The 1947 version in English is also available online, for example, at https://archive.org/details/pointlinetoplane00kand (accessed 2015-09-13).

[67] The original 1525 printing of *Underweysung der Messung* is available online, for example, at http://digital.slub-dresden.de/en/workview/dlf/17139/1/0/ (accessed 2015-09-13); see p. 7. An English-German bilingual facsimile edition is Albrecht Dürer: *The Painter's Manual*, translated with a commentary by Walter L. Strauss, 1977.

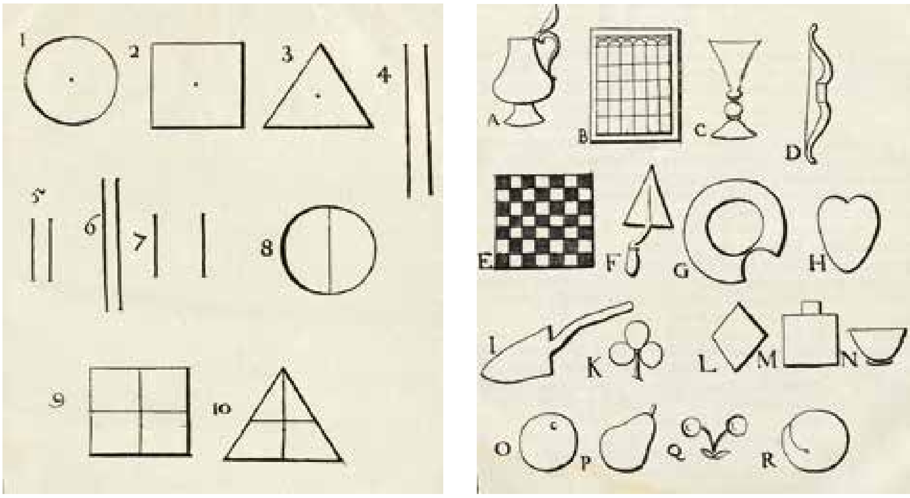


Figure 2.6 Images from the *Grondlegginge Ter Teekenkonst* by Gerard de Lairesse (1701).

and art theorist Gerard de Lairesse (16411711).[68] A few years later de Lairesse wrote another, even more influential and massive treatise on painting: *Het Groot Schilderboek*, “The Great Book of Painting”, which was first published 1707 in Amsterdam, and several editions thereafter, also in other languages.[69] In the first image[70] of his *Grondlegginge Ter Teekenkonst*, de Lairesse introduces a single point (A), one vertical line (B), a pair of converting lines (C), one horizontal line (D), and then three bending lines, the last of which looks like a hook (F), see Fig. 2.5 (right).[71]

In his second image, see Fig. 2.6 (left), de Lairesse depicts a circle, a square, and a triangle with a set of parallel lines, and then again an identically-sized circle,

[68] De Lairesse went blind in 1690 at the age of 49. He had to give up his successful career as a painter and concentrated thereafter on art theory. See, for example, https://www.rijksmuseum.nl/en/explore-the-collection/overview/gerard-de-lairesse (accessed 2015-07-13). *Grondlegginge Ter Teekenkonst* (1701) is available online at http://digi.ub.uni-heidelberg.de/diglit/lairesse1701bd1 (accessed 2015-07-13).

[69] Claus Kemmer, “In Search of Classical Form: Gerard de Lairesse’s ‘Groot Schilderboek’ and Seventeenth-Century Dutch Genre Painting”, in *Simiolus: Netherlands Quarterly for the History of Art*, Vol. 26, No. 1/2 (1998), pp. 87–115, available also online at http://www.jstor.org/stable/3780872. For an online version of the first 1707 edition, see for example, http://solo.bodleian.ox.ac.uk/OXVU1:oxfaleph015581140. The 1712 edition is available at https://archive.org/details/grootschilderboe00lair (all three 2015-09-15).

[70] The frontispiece with a novice, muse and Mother Nature is not counted here. In the next chapter, we will spend a little bit more time with Her.

[71] De Lairesse (1701), p. 3.

square, and triangle, this time divided into two, four, and four parts, respectively.[72] Further, in his treatise, de Lairesse presents common objects such as a window, a checkerboard, a small shovel, a bow, and some fruits, among others, to show how these shapes – or more precisely, their contours – can be approximated with the help of the aforementioned geometric forms, see Fig. 2.6 (right).[73] De Lairesse's *Grondlegginge* used geometry to show how the rich visual shapes seen in nature can be constructed from – or deconstructed to – a few basic geometric forms and lines. Dürer's *Underweysung*, on the other hand, was written at least to look like a mathematical treatise, even if it didn't make a difference between exact and approximate solutions and contained many inaccuracies and even some plain errors.[74] Dürer's approach could be described as "constructivist" when compared to de Lairesses's "reductionist" approach.

This "Euclidean" circle-square-triangle approach was, nevertheless, not universally adopted. For example, the famous English satirical painter and engraver William Hogarth (1697–1764) didn't put much weight on the importance of elementary geometric forms in his book *The Analysis of Beauty* (1753).[75]

Hogarth emphasized the role of a more dynamic and versatile S-curve, which he chose to depict in the title page of his book placed inside a glass pyramid with the word *VARIETY* on the plinth; see Fig. 2.7.[76] Nevertheless, the circle-square-triangle approach was again

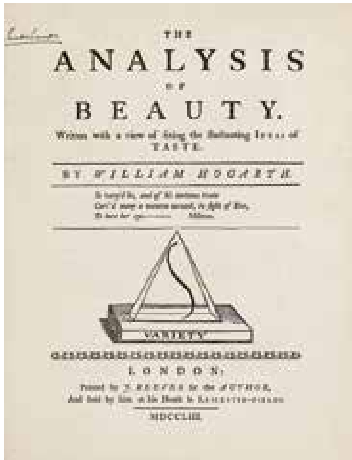


Figure 2.7 The title page from William Hogarth's *The Analysis of Beauty* (1753) depicts a smooth S-curve as a serpent inside a glass pyramid on a pedestal with the inscription "Variety".

[72] *Ibid.*, p. 7.

[73] *Ibid.*, p.12.

[74] Using only a compass and a ruler, constructing a regular heptagon (7-sided polygon), for example, is impossible in a precise mathematical sense. Dürer committed a most curious error when he placed (regular) heptagons and pentagons in larger circular shapes and stated that ten pentagons do not match to form a circular, closed dodecagon (10-sided polygon), even though they easily do. The 1977 editor Walter W. Strauss also comments on this strange error; see Dürer (1977), pp. 164–166. Strauss in turn refers to an article by Hermann Staigmüller: *Dürer als Mathematiker*, [in Programm des Königlichen Realgymnasiums in Stuttgart am Schlusse des Schuljahrs 1890/91], pp. 3–59, Stuttgart 1891, also available online at http://www.ub.uni-heidelberg.de/archiv/13288 (accessed 2015-09-17).

[75] William Hogarth, *The Analysis of Beauty*, 1753 is also available online, for example, at https://archive.org/details/analysisofbeauty00hoga (accessed 2016-03-08).

[76] Hogarth (1753), p. 1.

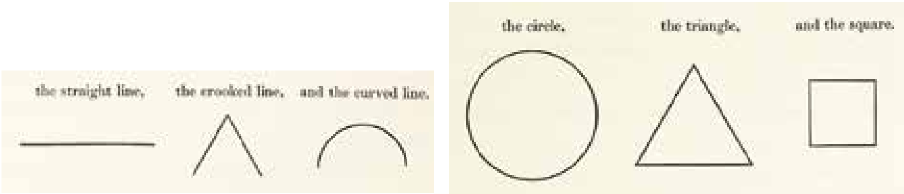


Figure 2.8 Two images from *The Natural Principles and Analogy of The Harmony of Form* (1842) by David Ramsay Hay.

clearly stated in the beginning of the 19th century by another English artist, William Robson (active *c*. 1800), who wrote in his *Grammigraphia; or the Grammar of Drawing* (1799): "Lines are Four: Perpendicular, Horizontal, Oblique, Curve. […] All the variety of appearances in nature are presented by a combination of these four lines, placed agreeably to Proportion and Position."[77] We can either consider a continuous one-dimensional line as a collection of infinite many zero-dimensional points, or as a static record of a dynamic trajectory of only one point moving in space. Robson favoured this latter approach, as he wrote further: "A Line is properly the continuation of a Point. A Point can proceed in four different ways only: Perpendicular, Horizontal, Oblique, Curve."[78] Four decades later David Ramsay Hay (1798–1866), a prominent interior and decorative designer in Edinburgh, wrote in 1842 in his book *The Natural Principles and Analogy of The Harmony of Form*: "But before proceeding to actual forms, it will be requisite to make a few observations upon lines; for by these, whether real or imaginary, all forms are represented. There are only three kinds of lines used in producing forms, and they are – the straight line, the crooked line, and the curved line. All varieties of form, however complex, all the similarity and dissimilarity that combine in the harmony of forms, are produced by these simple elements."[79] From these lines, Hay derives the three elementary forms: "As in sound and colour, so in form, there

[77] Robson (1799), pp. 40 and 44 at http://digi.ub.uni-heidelberg.de/diglit/robson1799, also quoted in David Brett, Drawing and the Ideology of Industrialization, in Dennis P. Doordan (ed.), *Design History: An Anthology*, Massachusetts: MIT Press, 1995, p.7. Brett, however, does not acknowledge the interruption of three pages in his use of this particular quote. The full title of Robson's book is *Grammigraphia; or the Grammar of Drawing: A System of Appearance, Which, by easy Rules, Communicates its Principles, and Shews How it is to be Presented by Lines; Distinguishing the Real Figure in Nature from the Appearance, or Shewing the Appearance by the Reality; Rendering Visual Observation more Correct and Interesting; and Proposing the Pleasure, and Universality of the Science*. [shews = shows]

[78] Robson (1799), p. 122.

[79] David Ramsay Hay: *The Natural Principles and Analogy of the Harmony of Form*, 1842, pp. 14–15. Also available online at: https://archive.org/details/naturalprinciple00hayd (accessed 2015-12-10).

are only three simple, primitive, homogeneous parts; and they are – the circle, the triangle, and the square", see Fig. 2.8 (right).[80]

Teachers and Teachers' Teachers

From Dürer with his point, straight, and curvy lines, and from de Lairesse, Robson, and Hay with their circles, squares, and triangles, we can draw a long and at times thin, but theoretically homogenous line not only to the teachings of Wassily Kandinsky, but also to Johannes Itten (1888–1967), Paul Klee (1879–1940), László Moholy-Nagy (1895–1946), and Oskar Schlemmer (1888–1943), among others, of the Bauhaus school. My intention in this thesis is not to present, and even less to prove, the existence of a continuous chain of artists or theorists leading from Dürer to Bauhaus. Nevertheless, I do want to show how this type of "constructivist" point-line-plane-approach to simple forms was widely used in theories and teachings of art and drawing well before modernism.

Johannes Itten, for example, studied art in Geneva with the Swiss painter Eugène Gilliard (1861–1921) for one semester in 1909 and later again after 1912.[81] Itten later recounted how such studies "using the elementary forms of the square, the circle, and the triangle represented an introduction into [his] non-objective designs of the years 1915 to 1919."[82] In these simple geometric forms, Itten found a basis for his later theories of art. Itten was grateful to Gilliard and expressed in his letter to Gilliard from 1914 that he considered the Geneva professor's teaching method outstanding because it "offers something fundamental that every artist should know, because you [Gilliard] present the elements, the basis, for all artistic work."[83] These geometric forms may have had a more or less prominent role in Gillard's pedagogical work, as Itten testifies, but they are not that salient in his rather traditional-looking paintings.[84] Gilliard also introduced Itten to a textbook on the fundamentals of design, *Méthode de Composition Ornamentale* (1905), written by Franco-Swiss decorative artist Eugène Grasset (1845–1917).[85]

Grasset was a pioneer in Art Nouveau design, and his book is more of a cornucopia of geometric and decorative forms and their endless variations than any sort of

80 *Ibid.*, p. 17.

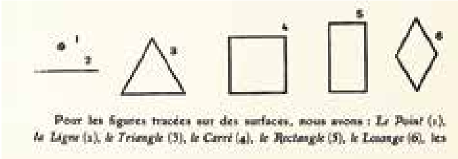
81 Rainer K. Wick, *Teaching at the Bauhaus*, 2000, p. 93.

82 *Ibid.* Wick refers to Johannes Itten's "Fragmente zu Leben und Werk" in Willy Rotzler (ed.), *Johannes Itten; Werke und Schriften*, 1978, p. 31.

83 Wick (2000), p. 93.

84 My opinion after a Google image search for the words "Eugène Gilliard" (search done 2016-05-25).

85 Wick (2000), p. 94.



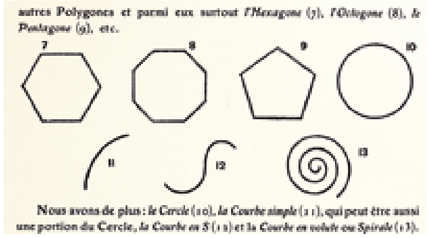


Figure 2.9 Two images from Eugène Grasset's *Méthode de Composition Ornamentale* (1905).

gospel of minimalism or even simplicity.[86] Grasset starts his 1905 treatise from the same geometric elements as so many artists before him – namely, from the point and the (straight) line. In addition to these visual elementary particles, he goes through the triangle, square, rectangle, rhombus, hexagon, octagon, and pentagon, with the laconic abbreviation "etc." for the rest of the regular polygons before introducing the circle. After these closed convex forms, Grasset mentions three more special cases of the line: the arc, the S-curve, and the spiral curve; see Fig. 2.9. These elementary forms would later become important not only in Itten's design theory but in Klee's and Kandinsky's theories of form as well.[87]

Itten's belief in the simple geometric forms was further strengthened when in 1913 he went to Stuttgart and studied there at the Academy with painter Adolf Hölzel (1853–1934).[88] Oscar Schlemmer and his lifelong friend Willy Baumeister (1889–1955) were also Hölzel's students. Hölzel stated that "simple forms" – by which he specifically meant the triangle, the square, and the circle – are "the best and most certain basis for artistic exploration".[89] A small catalogue book *Hölzel und sein Kreis* (Hölzel and his circle) was published in 1916 on the occasion of an exhibition of the same name.[90] The book contained an essay "Fragmentarisches" (Fragmentary remarks) by Johannes Itten in which he wrote: "The form most easily grasped and defined is geometrical, and its basic elements are the circle, the square, and the triangle. These form elements contain the seed of every possible form. Visible to those who can see, invisible to those who cannot."[91]

86 Grasset's extensive book of 900 pages is a two-volume set, available online at https://archive.org/details/mthodedecomposit01gras and [...] 02gras (accessed 2016-05-25).

87 Wick (2000), p. 94.

88 *Ibid.*

89 *Ibid.*, p. 95. Wick refers to Adolf Hölzel's "Einige aphoristisches Sätze aus einem demnächst erscheinenden Heft", in *Hölzel und sein Kreis*, 1916, also available online at https://archive.org/details/hlzelundseinkr00stutuoft (accessed 2016-06-01).

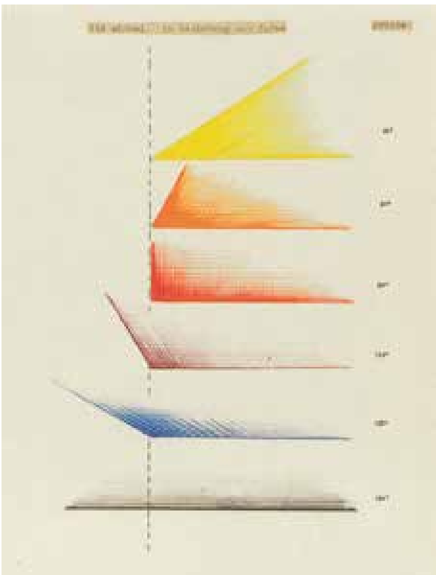
90 *Hölzel und sein Kreis* the exhibition and the book both included works by Baumeister, Itten, and Schlemmer, among other internationally less famous artists.

91 Wick (2000), p. 99, or see the original in *Hölzel und sein* Kreis (1916), p. 16.

Basic Forms and Basic Colours

Naturally, it was not only forms which interested artists of the time in theory and in practise, but also colours and their relations to the forms. Even further-reaching associations than just colours were also seen in simple forms. Itten, for example, wrote in 1916 in his diary about "form characters" in the following way: "Square: calm–death–black–dark–red, Triangle: severity = life–white–bright–yellow, Circle: zero to infinity = constant … calm … always blue."[92] And in 1917: "Square: horizontal–vertical, calm–rigidity, harmonious, Triangle: diagonal, meditation, inharmonious, Circle: formal, movement, harmonious."[93]

Itten's theory of the colour–angle correspondences was exactly the same as that of Wassily Kandinsky. The core idea shared by both of them was that sharper angles correspond to brighter and warmer colours, and as the angle grows, the corresponding colours turn darker and colder. A simple relation was introduced: acute angles 30°–60° = yellow–orange, right angle 90° = red, obtuse angles 120°–150° = violet–blue, and 180° = black.[94] See Fig. 2.10 (left).



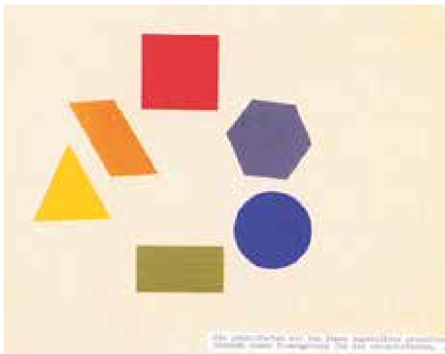


Figure 2.10 Two example of Kandinsky's students' exercises: Friedly Kessinger-Petitpierre, *the angles in relationship to colour*, 1929/30 (left), and Monica Ullman-Broner, *the primary colours with the primary forms assigned to them, attempt at giving form to the secondary colours*, 1931 (right).

[92] *Ibid.* p. 112, Wick refers to the Itten Diary no. 4 (October/Nov. 1916) in Eva Badura-Triska (ed.) *Johannes Itten: Tagebücher*, 1990, Vol. 2, p. 143.

[93] *Ibid.* p. 112, Wick refers to the Itten Diary no. 10 (10 November 1917) in Eva Badura-Triska (ed.) *Johannes Itten: Tagebücher*, 1990, Vol. 2, p. 372.

[94] See Kandinsky, *Point and Line to Plane*, 1947, pp. 72–74.

From these angles, the "characteristic colours" of the three primary forms were obtained: the triangle = yellow, the square = red, and the circle = blue, see Fig. 2.10 (right).[95] With a more stringent logic, the equilateral triangle, with its 60° angles, should have orange as its characteristic colour and not yellow, which was assigned to sharper angles. On the other hand, it makes sense that Kandinsky tried to find three "pure" colours which had as *different* a character as possible from each other; thus yellow, blue, and red were selected. A similar logical issue arises with the form of the circle, which has no angles at all. It seems that the circle was the best, or perhaps only, real option available with these rules as *the* third basic form if again as *different* characters as possible were sought. Kandinsky was also aware of the contemporary scientific study of the perception of forms. He knew about the school of Gestalt psychology, which had developed in Germany some decades earlier.[96] Rainer K. Wick (2000) writes: "According to an oral communication from Kurt Kranz on 19 January 1981 (Kranz was a student at the Bauhaus from 1930 to 1933), Kandinsky frequently mentioned in his course the names of the main representatives of the school of Gestalt psychology. He always made it clear, however, that the results of Gestalt psychology were merely a confirmation of his own earlier findings."[97] Wick also tells how Oscar Schlemmer, for example, commented on this "scientific" angle–colour correspondence theory in 1923, not without an ironic undertone: "Kandinsky's classes: scientifically strict study of colour and form. For example: the search for the three elementary colours that correspond to the three basic forms (triangle, square, circle). The decision was reached that yellow went with the triangle, blue with the circle, and red with the square–once and for all, so to speak."[98] Three years later Schlemmer added: "All the experts agree on the yellow triangle, but not on the others. Instinctively I always make the circle red and the square blue."[99]

Wassily Kandinsky used a special, short one-sided questionnaire to "test" his theory of how the three primary colours pair with the three primary forms. The students of his class were asked to fill three elementary forms with three elementary colours. The outlines of the forms, that is, the triangle, square, and circle, were printed

[95] *Ibid.*, pp. 74–75.

[96] See, for example, Mitchell G. Ash, *Gestalt Psychology in German Culture 1890–1967; Holism and the Quest for Objectivity*, 1995.

[97] Wick (2000), p. 222; note 77.

[98] *Ibid.*, p. 212.

[99] *Ibid.* Unfortunately I have not been able to find out who in 1938 chose the minimalistic logo and the tautological name still used by the German company Blaupunkt "Blue dot". See, for example, http://www.blaupunkt.com/en/company/success-story/ (accessed 2016-06-21).

Figure 2.11 An example of Kandinsky's famous questionnaire to the Bauhaus student. Note the "wrong" pairing of primary forms and colours in this particular case (left) and the "right" pairing used in the 2009 cover of a book *Bauhaus* by Jeannine Fiedler and Peter Feierabend (right).

at the top of the questionnaire, to be filled in, each with only one colour, either yellow, red, or blue (colours were mentioned in this order in the questionnaire); see Fig. 2.11 (left).[100] Given the fact that this "experiment" was conducted in a school where the three primary forms were taught to specifically correspond with the three primary colours, the results seems somewhat surprising.

Apparently, Oscar Schlemmer was not the only one at Bauhaus who did not intuitively connect colours with forms according to Kandinsky's theory. Art historian Magdalena Droste, for example, who has worked since 1980 at the Bauhaus Archive in Berlin reproduced one questionnaire with the "correct" answers, that is, yellow triangle, red square, and blue circle in her 1991 book *Bauhaus 1919–1933*, but no fewer than three questionnaires with *identical* "wrong" answers in the 2014 book *Vassily Kandinsky – teaching at the Bauhaus*.[101] One example of such a questionnaire with "wrong" answers filled in by student Karl Herman Haupt can be seen in Fig. 2.11 (left). It is noteworthy that somebody had even collected and preserved these "wrong" answers in the Bauhaus Archives. Perhaps this indicates that the spirit at

[100] Droste (1991), p. 86, and Droste (2014), pp. 34–35.

[101] Magdalena Droste, *Bauhaus 1919-1933*, 1991, p. 86, and Magdalena Droste (ed.), *Vassily Kandinsky - teaching at the Bauhaus*, 2014, pp. 34–35.

Bauhaus was not as uniform and dogmatic as the spirit in the surrounding society soon became.

At the same time, there were also trends of geometric abstract art in Russia, especially in the works of Kazimir Malevich (1878–1935), the founder of *suprematism*.[102] Often the ideas behind abstract and geometric works had not only aesthetic but also physiological, philosophical, and spiritual aspects.[103] Sometimes these ideas seemed effortlessly to cross over boundaries of different disciplines. In 1907, for example, the Russian neurophysiologist, physician, and painter Nikolai Ivanovich Kulbin (1868–1917) organized an artists' group in Saint Petersburg, which he named *Triangle: Art and Psychology*.[104] He chose the name "Triangle" because he had concluded from his neurophysiological research that artists impose not only their feelings but also geometric figures onto nature.[105] I am unaware of whether these artists' observations were related to his research on alcoholics.[106]

The Bauhaus school and its ideas had a tremendous impact and influence not only on modern art but most of all on design and architecture in the 20th century. At least one modern designer has written a complete trilogy about the square, the circle, and the triangle. Bruno Munari (1907–1998), an Italian artist, designer, and polymath inventor who worked in many fields of visual creativity, wrote three books about geometric basic shapes in 1960.[107] The trilogy was published as a

102 Kasimir Malevich, *The Non-Objective World*, 1959, originally published in 1927 in German as volume 11 in the Bauhaus books series; see Part II: Suprematism, p. 66: "Under Suprematism I understand the supremacy of pure feeling in creative art. To the Suprematist the visual phenomenon of the objective world are, in themselves, meaningless; the significant thing is the feeling […]" In this 1959 book, Malevich's first name is transliterated as Kasimir. The common English style nowadays, however, is Kazimir. I am grateful to Jyrki Siukonen for pointing out this fact.

103 See Wassily Kandinsky, *On the Spiritual in Art*, 1946 (German original published in 1911), available online, for example, at https://archive.org/details/onspiritualinart00kand (accessed 2016-07-07). There was also the famous 1986 exhibition and the book (edited by Maurice Tuchman) with the same name: *The Spiritual in Art: Abstract Painting 1890–1985*. See also the essay by the Finnish art historian Sixten Ringbom (1935–1992) "Art in 'The Epoch of the Great Spiritual'; Occult Elements in the Early Theory of Abstract Painting", in *The Journal of Warburg and Courtauld Institutes*, London, Vol. 29 (1966), p. 386–418 and his book *The Sounding Cosmos; A Study in the Spiritualism of Kandinsky and the Genesis of Abstract Painting*, 1970.

104 Lynn Gamwell, *Exploring the Invisible: Art, Science, and the Spiritual*, 2002, p. 104.

105 *Ibid*.

106 John E. Bowlt, *Russian Art 1875–1975: A Collection of Essays*, 1976, pp. 115–118, and https://monoskop.org/Nikolai_Kulbin (accessed 2016-07-07).

107 Two of them, the *Discovery of the Circle*, and the *Discovery of the Square*, were published in English in 1965, but he third one *The Triangle* was published by Corraini Edizioni, Mantova, Italy, and it came out in English only in 2007. Corraini Edizioni worked closely with Munari on many projects for many years.

single volume in 2015.[108] Bruno Munari contributed significantly by innovating in the fields of children’s books, toys, didactic methods, tactile and kinaesthetic learning, and in colourful, playful creativity in general.[109]

Many children’s toys use simple geometric forms and solids with an obvious function of teaching children to perceive and understand their different natures and characteristics in two and three dimensions; see Fig. 2.13. Such learning of shapes is a prerequisite for many skills in our modern society. One such skill is of course the ability to read and write. I believe it is not just a coincidence that so many letters in our western alphabets, whether Latin or Greek, are directly modelled after strongly articulated geometric forms, for example, I, O, Δ, V, Λ, Γ,L, Π, H, T, X, N, Z; see also Fig. 2.5 (left).

Figure 2.12 The 1960s trilogy by Bruno Munari published in 2015 as a single book (left), and the logo, which the Architects’ Council of Europe used until the same year (right).

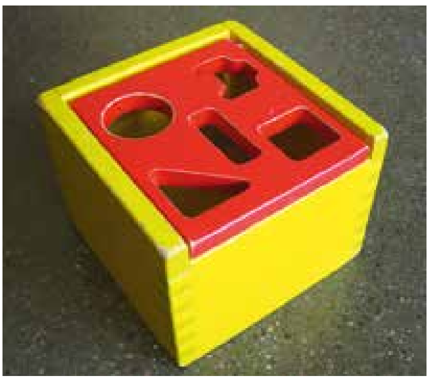


Figure 2.13 An example of a children’s toy for inserting simple geometric solids into a box with correspondingly shaped holes in the lid. A Finnish toy made of wood; Jukka lelut [Jukka toys], 1970s design, still in production (left), and a symbol from Microsoft Word for Mac for inserting shapes in a document, 2011 version 14.4.8 (right).

108 Published by Princeton Architectural Press, New York.

109 See, for example, Pierpaolo Antonello, Miroslava Hajek, and Luca Zaffarano (eds.): *Bruno Munari: My Futurist Past*, Silvana Editorale, 2013, Alessandro Colizzi: *Bruno Munari and the invention of modern graphic design in Italy, 1928-1945*, 2011, or Claude Lichtenstein and Alfredo Häberli (eds.): *Air Made Visible; A Visual Reader on Bruno Munari*, 2000. Munari himself wrote about two dozen books about design, visual communication and children’s books. Hundreds of scanned documents and texts related to his work are available online at http://www.munart.org (accessed 2016-02-25).

Fig. 2.14 The grey backsides of three traffic signs, each belonging to a different functional category to which their shape refers.

Basic geometric shapes are sometimes used directly as such to categorize information. With traffic signs, for example, one can deduce without even seeing the actual image whether it is a warning sign, a prohibitive (or restrictive or mandatory) sign, or a sign for some additional information.[110] On the front sides of the signs, only basic "pure" colours like red, yellow, blue, green, black, white, and middle grey are typically used. Lately, at least in Finland, a brown background colour has also been taken into use in connection with local specialities or countryside attractions worth a visit.

I admit that the comparison between the drawing manuals of the past centuries and the 20th-century modernistic view of art and design can feel somewhat anachronistic. Nevertheless, the bottom line here is that all of these artists and theorists promoted the same geometric tools to do the same work: to teach students of visual arts the basics of looking and drawing. The largest difference between them is of course that where for artists and theorists of the past centuries, the circle, square, and triangle were often no more than temporary scaffolds in producing the actual work, for modernistic purists, such rigid geometry provided more than occasionally the apparent essence of their final work.[111]

110 See [Vienna] Convention on Road Signs and Signals (1968), p. 4 for Article 2: (Road Sign Categories), and Annex 1, p. 30 for Section A: Danger Warning Signs (equilateral triangle), p. 37 for Section C: Prohibitory or Restrictive Signs (circular), p. 46 for Section E: Special Regulation Signs (squares or rectangles), and p. 50 for Section F: Information, Facilities or Service Signs (rectangular). The convention is also available online, for example, at http://www.unece.org/fileadmin/DAM/trans/conventn/signalse.pdf (accessed 2016-03-01).

111 One such example is the 1924 *Hochhausstadt* [high-rise city] plans by the German-American architect Ludwig Hilberseimer (1885–1967), which is perhaps one of the most dystopian visions the free world has produced in the realm of urban planning. Original perspective sketches are in the collections of the Art Institute of Chicago, see "Hilberseimer, Ludwig Karl" at http://www.artic.edu/aic/collections/artists/H (accessed 2016-07-15). Hilberseimer taught at the Bauhaus and wrote the introduction to Kazimir Malevich's 1927 book about suprematism. In Hilberseimer's defence, it must be said that even he himself later admitted his plans resembled more a *necropolis* than a *metropolis*. See Carsten Thau in Andreas Jürgensen and Karsten Ohrt (eds.), *The Mass Ornament*, 1998, pp. 60–61.

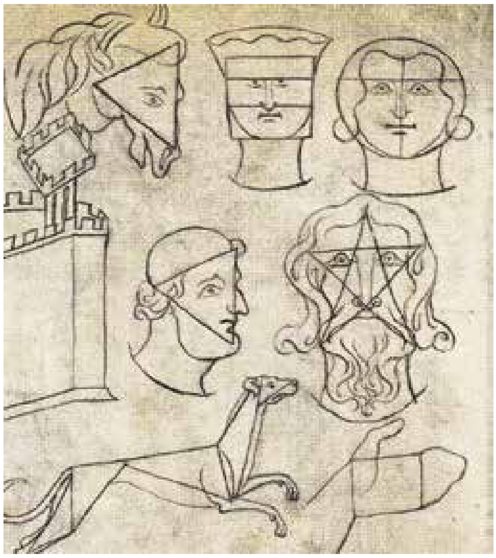


Figure 2.15 Villard de Honnecourt, studies of faces with geometric forms, *c.* 13th century.

Proportions of Man

Many artists have used geometric shapes to fuse, draw over or totally replace representations of animal or human figures especially in their studies and sketches. Well-known examples are the drawings of Villard de Honnecourt (active *c.* 1230), who most likely lived and worked mainly in Northern France, and Erhard Schön (*c.* 1491–1542), a follower of Dürer in Germany.[112]

In these examples, simple geometric shapes, especially the circle, triangle, and square are used to point out the most prominent forms and lines of, for example, humans and animals or sometimes to replace them partly or even completely. There's something doodle-like in many of Honnecourt's drawings, but nonetheless, they are carefully executed with good materials and preserved with care by all respects. The actual function of the drawings and even his profession are still disputed.

Often similar visual ideas seem to have been invented or observed or just recycled, and it is hard to know who actually invented some particular way of using geometric forms in the visual arts. The following Fludd-Weigel-Hogarth example is my own observation. A comparison can be made between three geometric studies

[112] Carl F. Barnes, *The Portfolio of Villard de Honnecourt (Paris, Bibliothèque nationale de France, MS Fr 19093): A New Critical Edition and Colour Facsimile*, 2007. Erhard Schön, *Unterweisung der Proportion und Stellung der Possen*, Nürnberg 1542; a 1920 reprint is available online at http://digi.ub.uni-heidelberg.de/diglit/schoen1920 (accessed 2015-07-14).

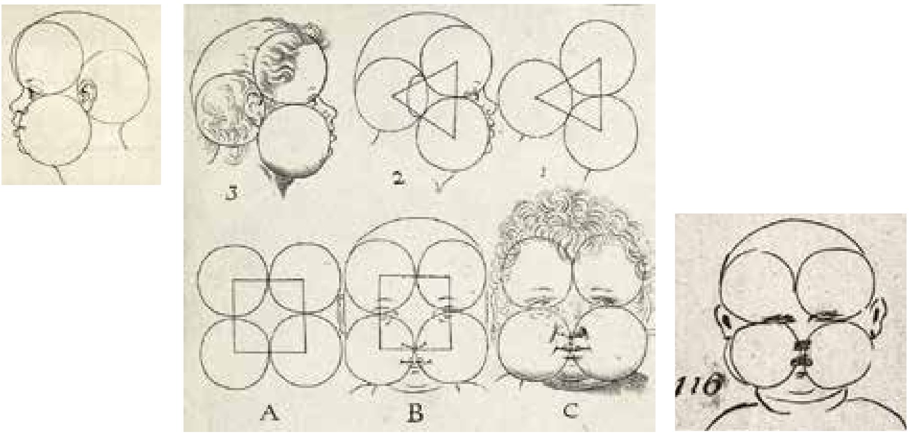


Figure 2.16 A detail of an image by Robert Fludd, 1617 (left), a detail of a model plate by Johann Christoph Weigel, 1705 (middle), and a detail from a book by William Hogarth 1753 (right).

of faces published in 1617 by Robert Fludd (1574–1637), an English physician with scientific and occult interests, in 1705 by German engraver and art dealer Johann Christoph Weigel [the Elder] (1654–1725), and in 1753 by William Hogarth in his book *The Analysis of Beauty*; see Fig. 2.16.[113]

In Hogarth's text, there is no reference to Weigel's earlier work, and in Weigel's work, there is no reference to Fludd's earlier work, but in the case of an another image in Fludd's book, there is an explicit reference to Dürer's work.[114] Some of Fludd's images differ also so distinctively in their style from the rest that it is reasonable to assume that they have different origins. Hogarth seems to have held in much higher esteem the ability of a form to "variate" than the "stability" of the forms. Interestingly, Hogarth also commented on his "baby-face" image exactly from this perspective: "[I]f we return back to infancy, we shall see the variety decreasing, till by degrees that simplicity in the form, which gave variety its due

[113] Fludd used the image of a head with three circles in the last page of his tractate *De usu arte geometrico in arte pictoria* [the use of geometry in pictorial arts] included in his 1617 *Utriusque cosmi maioris scilicet et minoris metaphysica* […], Vol. 1, which is available online at https://archive.org/details/utriusquecosmima01flud ; see p. 329. Johann Christoph Weigel [the Elder] published in 1705 in Nürnberg engraved model plates for artists entitled *Nützliche Anleithung Zur Edlen Zeichnung-Kunst*, also available online at http://digi.ub.uni-heidelberg.de/diglit/weigel1705 ; see the plate "ba". A high-resolution image of Hogarth's Plate II with the "baby-face" in the lower left corner is available online, for example, at http://www.metmuseum.org/art/collection/search/404839 (all three accessed 2016-03-08).

[114] Fludd, 1617, Vol. 1, p. 321.

limits, deviates into sameness; so that all the parts of the face may be circumscribed in several circles, as Fig. 116, plate 2."[115] [Here Fig. 2.16 right]

Sometimes the creatures with superposed circles, squares, triangles, and other geometric forms by Villard de Honnecourt are not just funny but almost verge on the plainly ridiculous. In many drawings of Villard de Honnecourt, the depicted geometric shapes and the faces look much more forced than they do in some studies by, for example, Albrecht Dürer; see Fig. 2.18. Also rather hard-shaped "box-heads" by Erhard Schön give a strong sense of the direction the head is facing, and expressing these directions seems to be their very essence, see Fig. 2.17.

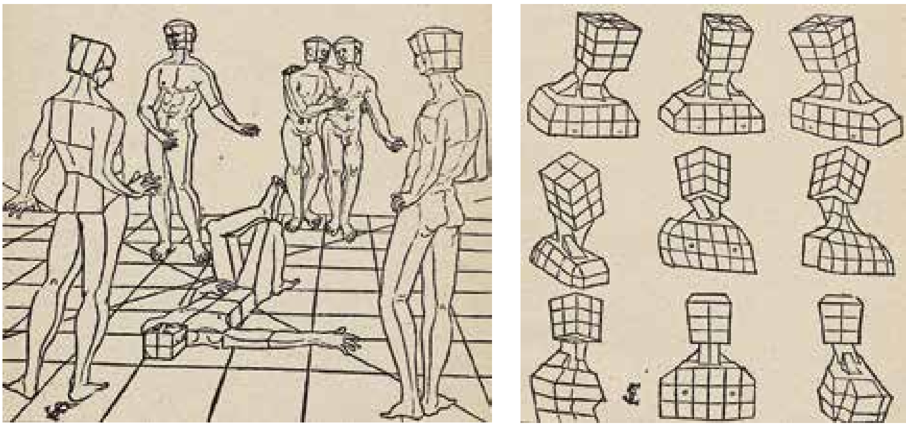


Figure 2.17 Erhard Schön, geometric studies of human forms and heads, 1542.

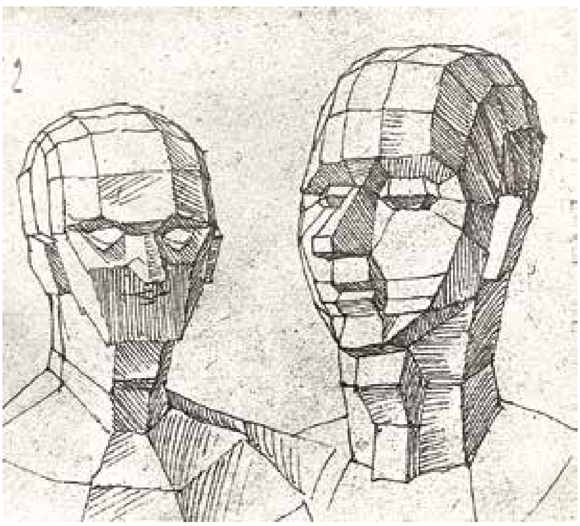


Figure 2.18 Albrecht Dürer, *Two Heads Divided into Facets*, from his sketchbook, 1519, Sächsische Landesbibliothek, Dresden.

[115] *Ibid*., p. 132

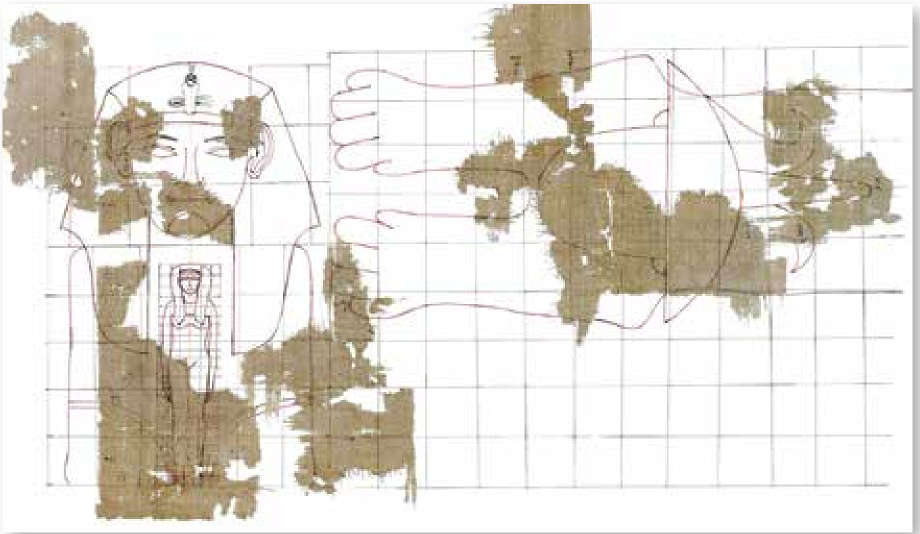


Figure 2.19 Fragments of the Berlin Sphinx Papyrus (brown) and a reconstruction of the grids they contain, Staatliche Museen zu Berlin, Ägyptisches Museum und Papyrussammlung.

In addition to producing these types of drawings with human (or animal) figures and simple geometric shapes as tools for study forms, directions, and volumes of bodies, they were also used to study the factual – and *measurable* proportions of the human (or animal) body. The influential art historian Erwin Panofsky (1892–1968) wrote about such a theory of proportions under the title "The History of the Theory of Human Proportions as a Reflection of the History of Styles" in his book *Meaning in the Visual Arts* (1955): "By a theory of proportions, if we are to begin with a definition, we mean a system of establishing the mathematical relations between the various members of a living creature, in particular of human beings, in so far as these beings are thought of as subjects of an artistic representation."[116]

In principle, the goal of the theory of proportions is thus simply to describe the physical facts about the relations of the various parts of the human body, but in practice, traditions and cultural conventions have had immense influence on how the body has been represented in visual art in different cultures. Because of these cultural differences, some images even with fairly similar geometric forms may have quite different intentions. Panofsky takes as his example the familiar square grid positioned atop some figurative image. In western art, this kind of a grid has been typically used to move and enlarge a sketch or a drawing into a larger scale, but in ancient Egypt the grid was primarily used to draw standardized figures

[116] Erwin Panofsky, *Meaning in the Visual Arts*, 1955, p. 56.

with very little variation. Human figures, for example, were always constructed with the aid of a square grid, where the height of the figure from the bottom of the feet to the hairline was first 18 units (the so-called "First Canon") and later, from the bottom of the feet to the upper eyelid, between 21 and 22 units (the so-called "Second Canon").[117] The famous Egyptologist and linguist Karl Richard Lepsius (1810–1884) was the first to make this observation of the two canons of construction grids used in ancient Egyptian art.[118]

In one instance where a sphinx with a statue of a goddess is sketched on papyrus, there are no fewer than three different – apparently even incommensurable – square grids used: one deemed proper for the lion body, one for the human face, and one for the goddess, that is, the human figure, see Fig. 2.19.[119]

The Greek sculptor Polykleoitos[120] (5th century BC) is a figure of paramount importance in the history of the theory of human proportions.[121] Danish architect and theorist Eivind Lorenzen, for example, has called the canon of proportions before Polykleoitos *archaic* and the canon of proportions after him *classical*.[122] One characteristic difference between them is the rigid symmetric position of the archaic sculptures compared to the asymmetric *contrapposto* of the classic sculptures. It is known that Polykleoitos wrote an influential treatise on the subject called simply *Canon*, and to illustrate his principles, he made a model statue *Doryphoros* "Spear-

117 *Ibid.*, pp. 57–66. Panofsky refers to Erik Iversen's book *Canon and Proportions in Egyptian Art*, London 1955. Unfortunately, I have been unable to get access to this study. A newer book by Gay Robins, *Proportion and Style in Ancient Egyptian Art*, 1994, has a rather critical presentation of some aspects of Iversen's work; see Robins (1994), pp. 40–56. Iversen's book is naturally referred to in another examination about ancient theories of human proportions, written by one of his countrymen, Eivind Lorenzen (1918–1994)*; Technological Studies in Ancient Metrology*, 1966, pp. 10, 20, and 64–69.

118 Lepsius led the Prussian expedition to Egypt and Nubia between 1842 and 1845 and oversaw the publication of its visual recordings in the 12-volume *Denkmäler aus Aegypten und Aethiopien* (1849–1859), a superbly illustrated folio-size work with nearly 900 coloured lithographs. The text part of the book was only finished in 1913, not during Lepsius' lifetime. This monumental work is also available online at the *Carl Richard Lepsius Project* site at http://edoc3.bibliothek.uni-halle.de/lepsius/info.html (accessed 2016-07-05). His metrological theories and observations were published in *Die Längenmasse der Alten*, 1884, available online at https://archive.org/details/dielngenmassede00lepsgoog (accessed 2016-03-21). Eivind Lorenzen wrote that the transition from the "First Canon" to the "Second Canon" happened during the XXVI dynasty (663–525 BC). Gay Robins, on the other hand, states that the transition started slightly earlier, during the XXV dynasty; see Lorenzen (1966), p. 60, and Robins (1994), pp. 35–37, 160 and 259.

119 Panofsky (1955); see p. 62 and Plate 18.

120 Πολύκλειτος is transliterated also as Polycletus, Polycleitus, Polyclitus, or Polyklitos.

121 Panofsky (1955), pp. 64–70.

122 Lorenzen (1966), p. 18.

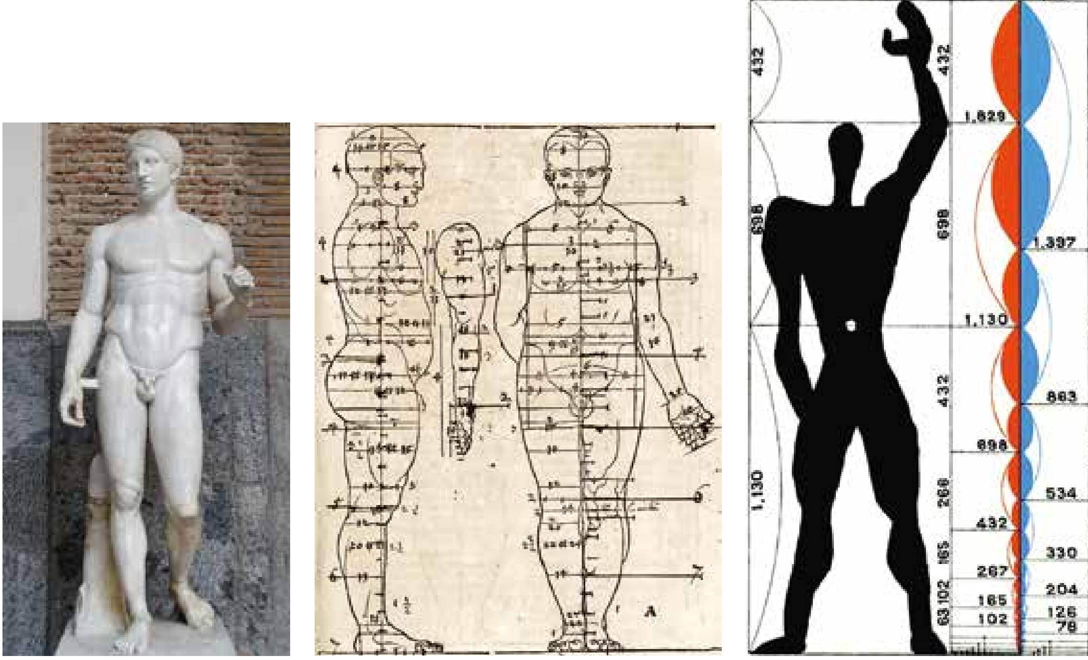


Figure 2.20 Polykleoitos, a model statue *Doryphoros* (*c.* 440 CB), a later Roman-period marble copy in the Naples National Archaeological Museum, Italy (left), Albrecht Dürer, a study of human proportions 1528 (centre) and the "Modulor man" by Le Corbusier (right).

Bearer", now known only from later Roman copies.[123] These classical rules for the proportions of an ideal nude male body have since been known simply as the *Polyclitan canon*.[124] See Fig. 2.20 (left).

Golden Ratio and *The Modulor*

Especially in ancient cultures, the question of human proportions was closely related not only to the practical nature of the anthropometric units, in which all measurements were typically taken, but also to a set of wider philosophical aesthetic questions about good proportions, harmony and beauty. Another, related metrological realm is the theory of architecture and the – real or hypothesized – measures and modules used in it. A classic in the field is the treatise *De architectura* by the Roman architect and military engineer Marcus Vitruvius Pollio (*c*. 85–*c*. 15 BC).[125] Another well-known but more recent contribution to the anthropometric

[123] Hugh Honour and John Fleming: *The Visual Arts: A History*, (7th ed.) 2010, p. 139. The word *Canon* comes from the Greek κανών [*kanon*] meaning (measuring) rod, rule, law, model, paradigm, or principle.

[124] Panofsky (1955), p. 64.

[125] In English the treatise is usually called *Ten Books on Architecture*. A 1914 edition is also available online, for example, at https://archive.org/details/vitruviustenbook00vitruoft and http://www.gutenberg.org/ebooks/20239 (both accessed 2017-03-29). For the latter, try scanning Fig. 5.7 (right) with a mobile phone.

theory of architecture and its measures is *The Modulor*, developed by the Swiss-born French architect Le Corbusier[126] (1887–1965) in occupied Paris during the Second World War and published soon after.[127] Le Corbusier's Modulor is based on the well-known ratio called the *golden* (or *divine*) *ratio* (or *section,* or *proportion),* or *golden number*, often marked with the Greek letter φ [phi] after the Greek sculptor Phidias (5th century BC), who is said to have used this proportion in his works.[128] One early and influential treatise on the subject is *De divina proportione* (On the Divine Proportion) by Luca Pacioli (*c*. 1445–1517), containing woodcut illustrations made after the drawings by his friend Leonardo da Vinci (1452–1519).[129]

The golden ratio divides a line ABC thus: A————B——————C into two parts, AB and BC, in such a way that for their lengths holds the equation AC : BC = BC : AB, meaning the length of the whole line, relates to the length of the larger part as the length of the larger part relates to the length of the smaller part. From this definition, the exact value of irrational φ can be calculated as ½ $(1+\sqrt{5})$, or approximately 1.618034. The golden proportion appears inherently in the regular pentagon and in many proportions of plants, and it has been excessively used, studied and referred to in art during last hundred years[130], but I will refrain myself from entering the genre further. The golden proportion touches the subject of my study but is not the focus.

126 Le Corbusier's original name was Charles-Édouard Jeanneret-Gris.

127 *The Modulor: A Harmonious Measure to the Human Scale Universally applicable to Architecture and Mechanics*, 1954. The French original appeared in 1948.

128 Mario Livio, *The Golden Ratio*, 2003, pp. 1–5. In more technical mathematical literature, the golden ration is marked with Greek letter **τ** [tau] after the word **τέμνειν** [témnein] "to cut". A French book by the Romanian diplomat, historian, mathematician, philosopher, poet and, quite modestly, Prince of Moldavia, Matila C. Ghyka (1881–1965) is often met in the literature dealing with the golden section: *Le nombre d'or,* (the golden number), two volumes; *Tome I: Les Rythmes, Tome II: Les Rites*, 1931.

129 *De divina proportione* was first printed in 1509 in Venice, and it is also available online, for example, at https://archive.org/details/divinaproportion00paci (accessed 2017-03-29).

130 Livio (2003), pp. 75–123 and Appendix 2 (pp.256–257) in his book. An influential group of French artists, poets and critics, for example, even named their collective, most active around 1911–1914, as Section d'or "golden section" after the proportion. The group included, among others, the three Duchamp brothers: Marcel Duchamp (1887–1968), Raymond Duchamp-Villon (1876–1918), and Jacques Villon (1875–1963), and the painters Albert Gleizes (1881–1953), Jean Metzinger (1883–1956), and Fernand Léger (1881–1955). See John Golding, Cubism: A History and an Analysis, 1988 (1959), pp. 10–16 and pp. 168–183.

Canon 60

In addition to the Modulor, there is another interesting but internationally perhaps less-known system, *Canon 60*, which is a theory of measures and proportions developed by the Finnish architect Aulis Blomstedt (1906–1979).[131] The name "Canon 60" is a proper one in at least two ways: Blomstedt published his studies from *c*. 1960 onwards, and the number 60 itself plays a central role in the theory because of its convenient divisibility properties. The number 60 is divisible by 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. In his works, Aulis Blomstedt always preferred round values and integers instead of fractional systems, which the inherently irrational golden section produces, for example, in Le Corbusier's Modulor. By triplicating the 60 cm module, one also gets a nice round approximation of the height of an adult male; see Fig. 2.21 below.

Figure 2.21 Aulis Blomstedt, modular variations of the 180 cm measure, published in the Finnish *Arkkitehti* magazine in 1957.

[131] In English, there is an essay by Juhani Pallasmaa: "Man, Measure and Proportion – Aulis Blomstedt and the tradition of Pythagorean harmonics", published in *Acanthus 1992, The Art of Standards / Standardien taide*, 1992. In Finnish, there is an excellent book written by Helena Sarjakoski: *Rationalismi ja runollisuus; Aulis Blomstedt ja suhteiden taide* [Rationalism and the poetic; Aulis Blomstedt and the art of the proportions], 2003.

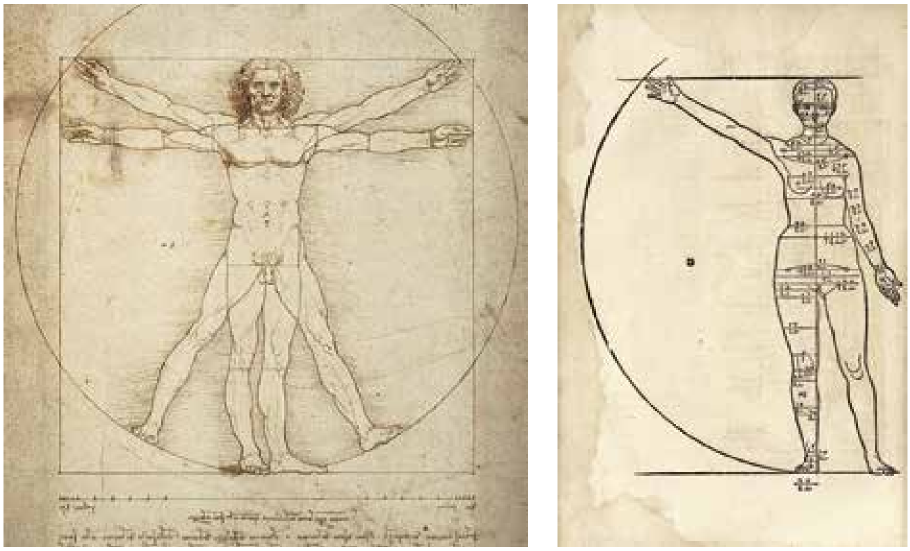


Figure 2.22 Leonardo da Vinci, *Vitruvian Man*, a drawing made around 1490 (left) and a study of the female proportions by Albrecht Dürer, 1528 (right).

The Vitruvian Man and The Dürerian Woman

One example of studies of human proportions done by an artist is Albrecht Dürer's treatise [*Hierinn sind begriffen*] *Vier Bücher von Menschlicher Proportion*[132], printed in 1528 in Nuremberg. The influence of Dürer's books can be noticed in several following centuries in the studies of geometry, perspective and human proportions.[133] Dürer's book is a detailed study of human proportions based on his own observations and the theories of Vitruvius.[134] In addition to Dürer, the rediscovered texts by Vitruvius had already inspired Leonardo a few decades earlier to draw a study of the human proportions containing two simple geometric forms, that is, the square and the circle: see Fig. 2.22 (left). Even if Leonardo's drawing

[132] See, for example, https://archive.org/details/hierinnsindbegri00dure (2015-08-12).

[133] One example of such a treatise is *Des Circkels und Richtscheyts* (1564) by the German goldsmith and painter Heinrich Lautensack (1522–1590), available scanned online at http://digi.ub.uni-heidelberg.de/diglit/lautensack1564 (accessed 2015-09-15).

[134] Erwin Panofsky: *The Life and Art of Albrecht Dürer*, 1955 (1943), pp. 261–268. For the Vitruvius texts in question, scan Fig. 5.7 (right) with your mobile phone.

Vitruvian Man (*c*. 1490) is basically only an anatomical study of a single human body, the actual work has invoked much deeper feelings in countless viewers throughout the past centuries, and nowadays, the *Vitruvian Man* is perhaps the most famous drawing in the world.[135]

The example human Leonardo depicted was a fit adult man in his forties or fifties. It can be argued whether Leonardo revealed here also his own *ideal* of human body. Dürer presented with precise measurements in his book a great number of different types of people: adults and children, thick and thin, male and female. It has to be added here, that Leonardo also depicted all kinds of different people in his drawings and paintings. Leonardo's iconic drawing seems to reflect not only man's relation to himself but also to the whole cosmos. There's something bold and existential in his presence. He looks neither humble nor arrogant; he simply *is*. And in what a peculiar state of being he is: standing immobile yet moving his hands and feet. He is at rest and in action at the same time.[136] He seems to have the ability, the will and the determination to figure out not only himself, but also the grand book of the universe.

[135] Paintings excluded here, but even if they were included, we would not get that far from Leonardo if we were to select "the most famous painting" in the world.

[136] Or, in the terminology of modern physics, is he in a quantum mechanical "superposition"?

3 | The Grand Book and its Frozen Ciphers

In this chapter I address two well-known metaphors of nature: the living feminine persona called Mother Nature and "the grand book of nature", written in an unusual language consisting of ciphers or strange characters, which Galileo Galilei considered to be simple geometric forms. I will inspect more closely two cases in which Johannes Kepler interpreted natural phenomena as being based on such geometric characters: the shapes of the orbits of the planets and snowflakes. Finally, with the help of the snowflake, I will briefly explain the concepts of *emergence* and *self-similarity*.

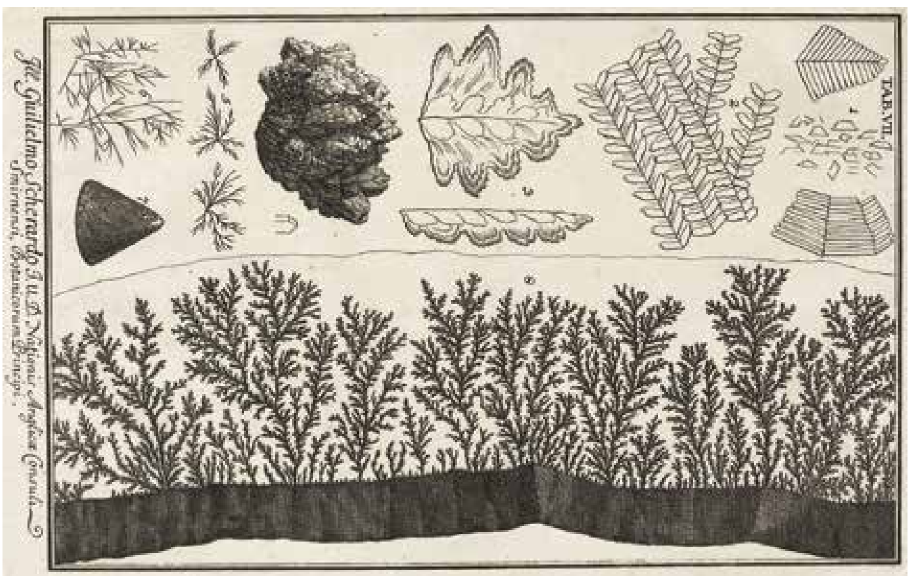


Figure 3.1 Plant-like structures inside stones depicted by the Swiss scholar Johann Jakob Scheuchzer (1672–1733) in his book *Herbarium diluvianum*, 1723 (first edition 1709).

Friedrich von Hardenberg (1772–1801), better known by his pen name Novalis, was one of the central figures in the literary branch of German Romanticism.[137] In the opening words of his book *The Novices of Sais*, Novalis refers to many living and non-living entities, forms and phenomena that are available for our unaided eyes:

> *Various are the roads of man. He who follows and compares them will see strange figures emerge, figures which seem to belong to the great cipher which we discern written everywhere, in wings, eggshells, clouds and snow, in crystals and in stone formation, on ice-covered waters, on the inside and outside of mountains, on plants, beasts and men, in the lights of heaven, on scored disks of pitch or glass or in iron filings round a magnet, and in strange conjunctures of chance. In them we suspect a key to the magic writing, even a grammar, but our surmise takes on no definite forms and seems unwilling to become a higher key.*[138]

On the one hand, Novalis' examples refer to things that have nearly always been available to human perception. On the other hand, scored disks of pitch or glass and iron filings round a magnet refer to things which exist partly via technical innovations of human culture. But even if Novalis mentions materials produced by human culture, he is not talking about artefacts of human culture here. He is referring not only to the variety of forms, figures and phenomena discerned in visible nature but foremost, he is referring to "the magic writing" and "the great cipher" to which these figures seem to provide a key.

The title of Novalis' romantic, heavily philosophical and mystical book refers to the story mentioned by Plutarch (*c.* 40–120) in his *De Iside et Osiride*, a treatise of Egyptian mythology.[139] The legend tells about a statue of the goddess Neith, located in the Egyptian city of Sais.[140] Neith was assimilated with Egyptian Isis and

[137] Novalis may not have been as influential in the literary circles of his own time as Goethe, Schiller, or the brothers Friedrich and August Wilhelm Schlegel were, but the death of his young fiancée Sophie at the age of 15 and his own early death caused by tuberculosis a few years later makes him an example of a romantic poet *par excellence*. See, for example, Ernst Behler, *German Romantic Literary Theory*, 2005, or Manfred Frank, *The Philosophical Foundations of Early German Romanticism*, 2003, pp. 151–176: 'On Novalis' Pivotal Role in Early German Romanticism.'

[138] Novalis, The Novices of Sais (1798), 2005, p. 3. Paul Klee did illustrations for this book in 1927, and Ralph Manheim translated it in English in 1949. Manheim was also the translator of the voluminous Paul Klee Notebooks Volume 1: The thinking eye, 1961.

[139] *Plutarch's De Iside et Osiride,* edited with an introduction, translation and commentary by J. Gwyn Griffiths, 1970, p.131.

[140] Written also as Saïs.

later with Greek Athena, and they in turn were considered as personifications of nature. Already in antiquity it was an old metaphor that nature personified kept her hidden from us. Plutarch tells: "At Saïs, the seated statue of Athena, whom they identified with Isis, bears this inscription: 'I am all that has been, that is, and that shall be; no mortal has yet raised my veil.'"[141]

At this point it must be mentioned that in ancient Greece, especially in the area around the city of Ephesus, the goddess Artemis was assimilated with personified nature, and thus with Isis.[142] Hence, the more accurate phrase *Artemis of Ephesus* is often used to make clearer the distinction from the other forms of the cult – or artworks related to it. By the mid-seventeenth century the image of the multi-breasted goddess Isis-Artemis as Mother Nature had become standard iconography in Western art, flourishing especially in the frontispieces of books dealing with the study of nature.[143] The use of the Isis-nature metaphor was especially popular during the Romantic era, coincidental with the time of the real *Egyptomania*.[144]

Not only Novalis but, for example, Friedrich von Schiller (1759–1805) in 1795 also wrote a poem with precisely the same Isis-statue theme: *Das verschleierte Bild zu Sais* "The Veiled Image at Sais".[145] Also Johann Heinrich Jung-Stilling (1740–1817)

[141] Pierre Hadot, *The Veil of Isis, An Essay on the History of the Idea of Nature*, 2008, p. 265. Later the influential Neo-Platonist philosopher Proclus (AD 412–485) added the words "The fruit of my womb was the sun" to the end of the text. See Proclus, *The Six Books of Proclus* […], translated from the Greek by Thomas Taylor, 1816; Vol 2, book VII, pp. 169, also available online at http://catalog.hathitrust.org/Record/002240527 (accessed 2016-02-29). Also, the eminent Egyptologist Jan Assmann has analyzed the meaning of this inscription in his book *Moses the Egyptian; The Memory of Egypt in Western Monotheism*, 1998, pp. 118–119.

[142] See, for example, Hadot (2008), chapter 19: 'Artemis and Isis', pp. 233–243.

[143] One illuminating example of such a usage is found in Athanasius Kircher's *Mundus Subterraneus* […], Vol. 2, Amsterdam, 1678, which is also available online, for example, at http://digi.ub.uni-heidelberg.de/diglit/kircher1678bd2 (accessed 2016-02-26).

[144] One such example is the journal Isis, eine Encyclopädische Zeitschrift, vorzüglich für Naturgeschichte, vergleichende Anatomie und Physiologie, published from 1816 to 1848 and founded by Lorenz Oken (1779–1851), a German naturalist, botanist and biologist and one of the leaders of the Naturphilosophie movement in Germany. The same Isis–Nature metaphor is behind the name of the journal Isis, An International Review devoted to the History of Science and its Cultural Influences, published since 1912 and founded by George Sarton (1884–1956), a Belgian-American chemist and historian who is considered the founder of the modern discipline of the history of science. At the other end of the literary spectrum, we find H. P. Blavatsky (1831–1891) and her Isis Unveiled; A Master-Key to the Mysteries of Ancient and Modern Science and Theology, 1877. It is one of the central works in the esoteric Theosophical movement, which she helped to form and where she played a prominent role.

[145] An English translation of Schiller's poem is available online, for example, at http://www.bartleby.com/270/12/98.html (accessed 2016-02-18). The aforementioned Assmann has also made a study of Schiller's poem in his book Das verschleierte Bild zu Sais: Schillers Ballade und ihre griechischen und ägyptischen Hintergründe, 2011.

Figure 3.2 *Artemis of Ephesus,* a Roman copy of the Greek original, Galleria delle Candelabri, Vatican Museums (left), and Markus Rissanen, *Mother Nature – Mother Night*, [M140], 110x100cm, acrylic on canvas, 2012 (right).

used the same theme in the second part of his novel *Das Heimweh* "Homesickness", originally published in four parts in 1794–1796.[146]

Wolfgang Amadeus Mozart (1756–1791) is known to have visited the Temple of Isis at Pompeii, Italy in 1769, just a few years after it was unearthed.[147] His visit is considered to have inspired him later to compose *The Magic Flute* (1791), which contains Sarastro's aria "O Isis und Osiris". The opera is set in ancient mystery-laden Egypt. Saïs and its lore inspired also others; the French composer Hector Berlioz (1803–1869) included the oratorio *L'Arrivée à Saïs* "The Arrival to Saïs" in his work *L' Enfance du Christ* "The Childhood of Christ" (1854), in which he imagined the Holy Family visiting ancient Sais.[148] Ludwig van Beethoven (1770–1827) had a framed version of the Isis-Artemis inscription on his desk.[149] Immanuel Kant also valued this inscription especially highly as he said in his *Critique of Judgment* (1790): "Perhaps nothing more sublime was ever said and no sublimer thought

146 See p. 147 in the first (1794) edition, or p. 107 in the 1826 edition, both available scanned via Google books (accessed 2016-02-29). Jung-Stilling even wrote *Der Schlüssel zum Heimweh*, (A Key to the '*Heimweh*'), and this appeared as the fifth part also in 1796.

147 Matheus F. M. van den Berk, *The Magic Flute; Die Zauberflöte; an Alchemical Allegory*, 2004, p. 450: 'Mozart visits the Isis temple in Pompeii'.

148 D. Kern Holoman, *Berlioz*, 1989, p. 454.

149 Erik Hornung, *The Secret Lore of Egypt: Its Impact on the West*, 2001, p. 134.

ever expressed than the famous inscription on the Temple of Isis (Mother Nature): 'I am all that is and that was and that shall be, and no mortal hath lifted my veil.'"[150]

The Swiss artist Henry Fuseli [Johann Heinrich Füssli] (1741–1825), who spent a great part of his life in Britain, made the illustrations for the book *The Temple of Nature* (1803), including also the frontispiece depicting Mother Nature, or Isis-Artemis, with her multiple breasts.[151] The book was written by Erasmus Darwin (1731–1802), an English physician, natural philosopher and a grandfather of the famous Charles Darwin (1809–1882). As we can see with these numerous Saïs-Isis-Artemis related examples, Novalis' book is only the tip of the iceberg, or, in this more oriental context, shall we say, the apex of the pyramid.

The Grand Book of Nature

But what does Novalis mean by saying that some forms are examples of the great cipher written everywhere, that is, in nature? What does he mean by the magic writing and its grammar? What is the key to this great cipher he is talking about? Novalis did not provide the solution to this cipher in work, but another thinker was more straightforward in his opinion about the matter two hundred years earlier. In fact, he had boldly stated that we not only have access to the key and grammar of nature but that we can actually read nature, that is, the universe, like an open book:

> *Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics and its characters are triangles, circles, and other geometrical figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.*[152]

[150] Immanuel Kant, Critique of Judgment, translated with introduction and notes by J. H. Bernard, 1914; see Kant's last footnote of §49; 160. This 1914 edition is also available online at https://ebooks.adelaide.edu.au/k/kant/immanuel/k16ju/index.html (accessed 2016-02-26). The Greek word πέπλος [peplos] is most often translated as "veil", whereas another tradition of translation – and interpretation – translates it as "mantle" or "tunic". There is certainly a difference whether only the veil or the whole tunic is lifted. The sexual nature of the latter is well presented by J. Gwyn Griffiths, who in his book (1970), p. 284 cites an old magic papyrus: "[Isis], pure virgin, give me a sign of the fulfilling (of the charm), lift the sacred mantle, shake thy black Destiny." I suspect Kant was not referring to this more carnal interpretation of the inscription in his judgement.

[151] Erasmus Darwin, *The Temple of Nature; or, the Origin of Society,* 1803, also available online, for example, at https://archive.org/details/templeofnatureor00darw (accessed 2016-02-25).

[152] Galileo Galilei: *Il Saggiatore* (the Assayer), 1623, reprinted in (1957), pp. 237–238.

Here we have perhaps the most famous quotation from one of the most famous scientists of all time, namely Galileo Galilei.[153] He was a champion of modern science, a practice that relies more on observations and experiments than on traditions or authorities. Like Novalis, Galileo also talks of something we can see or "read" in nature, and both of them talk about writing, language and grammar. But unlike Novalis, Galileo does not write about *perceivable* entities in nature. In a way, they both seem to talk about the same thing, or at least closely related things, even with similar words, but from two different perspectives. Just as Novalis wrote about how nature *looks*, Galileo wrote about the basic forms we use to understand how nature *works*.

Let me make a few short remarks about the words of Galileo before going further. Firstly, the poetic metaphor of the universe as a book was not Galileo's original invention as the literary history of the expression can be traced back to St. Augustine (AD 354–430).[154] Secondly, I interpret "nature" in the broadest possible sense to mean the same as "universe". Thirdly, Galileo used the word "philosophy" when he talked about science or our rational knowledge of the world in general. The terms *philosophy* and *natural philosophy* were widely used before the word *science* gradually replaced them. The way in which William Shakespeare (1564–1616), for example, used the word in the well-known lines of *Hamlet*, when the protagonist and his friend face a ghost, is telling: "There are more things in

[153] The even more famous words "*Eppur si muove*" (And yet it moves) by Galileo after he was forced by the Inquisition to renounce his belief that the Earth moves around the Sun are almost certainly an apocryphal invention. Probably Galileo was not the first scientist to use the recently invented telescope to make astronomical observations, but he was the first one to *publish* such observations in his book *Sidereus Nuncius* (1610).

[154] St. Augustine believed that we could read the divine message of God in two 'books': in nature and in the Bible. Among many others, Bonaventura (1217–1274), St. Thomas Aquinas (1225–1274) and Paracelsus (1493-1541) used the same metaphor before Galileo. A comprehensive survey of the metaphor and its early cultural history in the Western world can be found in Arjo Vanderjagt and Klaas van Berkel (eds.), *The Book of Nature in Antiquity and the Middle Ages,* 2005. Carla Rita Palmerino studies specifically Galileo and the book metaphor in her essay "The Mathematical Characters of Galileo's Book of Nature" in Arjo Vanderjagt and Klaas van Berkel (eds.), *The Book of Nature in Early Modern and Modern History,* 2006. The same metaphor was also used in medieval Islamic culture between nature and the Quran. See, for example, Seyyed Hossein Nasr, *Islamic Art and Spirituality,* 1987, note 6 in p. 39 and p. 60. Similar sentiments were expressed by the English author and physician Sir Thomas Browne (1605–1682), who wrote in 1642: "The Finger of God hath left an inscription upon all his works, not graphical or composed of Letters, but of their several forms, constitutions, parts and operations, which, aptly joined together, do make one word that doth express their natures. By these Letters God called the Stars by their names; and by this Alphabet Adam assigned to every creature a name peculiar to its Nature." *Religio Medici,* The Second Part, Chapter II, p. 91 in the 1902 edition, which is also available online at https://archive.org/details/religiomedicioth00browrich (accessed 2016-09-29). Marjorie Hope Nicolson quotes Sir Thomas Browne several times in her *The Breaking of the Circle; Studies in the Effect of the 'New Science' upon Seventeenth-Century Poetry,* 1960; this passage is found in p. 38.

heaven and earth, Horatio, than are dreamt of in your philosophy."[155] The title of Isaac Newton's (1642–1727) famous book *Principia* (1687) also reads in totality as *Philosophiæ Naturalis Principia Mathematica*, Latin for "Mathematical Principles of Natural Philosophy".[156] And from Newton we come back to Novalis, Galileo and to the point of the matter. Both scientists shared the opinion that nature, or the universe, is comprehensible to us humans, and its laws are written in the language of mathematics. In addition, Novalis, who had received a practical university training in geology and had worked as a manager and engineer in salt mines in Saxony, referred often in his writings to chemistry, physics and in particular, to mathematics.[157] Often this was done in a theistic tone, as in one of his last fragments (1799–1800), in which he wrote: "Cannot God also reveal himself in mathematics, as in every other branch of learning?"[158] For a moment I shall try to leave God and the gods aside and head back to geometry and its figures.

The Pedigrees and Patents

What kind of language and what kind of principles was Galileo talking about when he talked about mathematics and geometry? What did he mean by saying that the characters of this language are triangles, circles and other geometric figures? How could we describe nature and its principles with geometric figures when in normal life we almost never see such forms in nature with our unaided eyes, with the possible exception of the circular images of the Sun and the full Moon in the sky? Are some forms "better" or even more "noble" than others? Galileo, who showed not only witty but also biting sarcasm in his writings, gave his opinion in his *Il Saggiatore* (1623):

[155] William Shakespeare: *Hamlet*, Act 1, Scene 5.

156 See, for example, *The Principia: Mathematical Principles of Natural Philosophy by Isaac Newton*, a new translation by I. Bernard Cohen and Anne Whitman with the assistance of Julia Budenz, University of California Press, Berkeley, California 1999.

[157] Especially in his unfinished and fragmentary encyclopedic project *Das Allgemeine Brouillon* (The General Draft), *Mathematische Fragmente*, and *Mathematischer Heft* (all three; winter 1798–1799), all include some poetic, philosophical, and mystical treatments of mathematical equations. See, for example, Novalis, *Notes for a Romantic Encyclopaedia; Das Allgemeine Brouillon*, translated and edited with an introduction by David W. Wood, 2007, entry no. 933, p. 166: "God is now [sometimes] $1 \cdot \infty$, now [sometimes] $1/\infty$, now [sometimes] 0." For a general study of the theme, see for example, Gabriel Rommel, "Romanticism and Natural Science", pp. 209–227 in Dennis Mahoney (ed.), *The Literature of German Romanticism*, 2004. A more specific study of the subject is available in German; Franziska Bomski, *Die Mathematik im Denken und Dichten von Novalis*, 2014.

[158] *Novalis: Philosophical Writings*, translated and edited by Margaret M. Stoljar, 1997, fragment no. 38, p. 162.

Before I proceed let me tell Sarsi that it is not I who want the sky to have the noblest shape because of its being the noblest body; it is Aristotle himself, against whose views Sig. Guiducci is arguing. For my own part, never having read the pedigrees and patents of nobility of shapes, I do not know which of them are more and which are less noble, nor do I know their rank in perfection. I believe that in a way all shapes are ancient and noble; or to put it better, that none of them are noble and perfect, or ignoble and imperfect, except in so far as for building walls a square shape is more perfect than the circular, and for wagon wheels the circle is more perfect than the triangle.[159]

Galileo had immersed himself in a serious dispute about the nature of comets with Orazio Grassi (1583–1654), a Jesuit priest, architect and astronomer. In 1619, Grassi published a treatise, *Libra Astronomica ac Philosophica qva Galilaei Galilaei Opiniones de Cometis* […] *examinatur* […], under the pseudonym Lothario Sarsi Sigensano, an anagram of his Latin name.[160] As its title reveals, the book was presented as a scale which examines Galileo's opinions of comets and finds them wanting. Galileo gave his literal response only four years later (1623), but in a tone no less sharp. The title *Il Saggiatore* "the Assayer"[161] means someone who tests or examines something carefully in order to assess its nature, quality or value. The word "assay", which Galileo used, referred not only to a researcher doing observations, tests and experiments, but also ridiculed the very word "Libra" (scale, balance), which Grassi had used. To "assay" was an allusion to precision scales used for weighing gold and other precious materials in tiny amounts, whereas "Libra"

[159] In *Discoveries and Opinions of Galileo*, 1957, pp. 262–263. The aforementioned Sir Thomas Browne, on the other hand, held the firm opinion that there *is* one particularly noble form, the *quincunx*, i.e. "an arrangement of five objects with four at the corners of a square or rectangle and the fifth at its centre, used for the five on a dice or playing card, and in planting trees." Sir Thomas even wrote a complete book of this quincuncial arrangement only: *The Garden of Cyrus, or, The Quincunciall, Lozenge, or Net-work Plantations of the Ancients, Artificially, Naturally, Mystically Considered* (1658). A later 1927 facsimile edition is also available online at https://archive.org/details/hydriotaphiaurne00browuoft (accessed 2016-10-26), in the original pagination; from p. 86 onwards.

[160] Lotharius Sarsis Sigensanus = Horatius Grassius Salonensis [Orazio Grassi]. See, for example, J. L. Heilbron: *Galileo*, 2010, glossary of names, p. 384, or Pietro Redondi: *Galileo; Heretic*, 1987, Fig. 5.30d ike Platois there a mistake here?, hmätapaamisissa.t ample, at s by uvius. e jo ihmisiä.. le verkostuneita projekteja.ä eri johtoryhmätapaamisissa.t

[161] The first half of the title reads in English as *THE ASSAYER, In which with a most just and accurate balance there are weighed the things contained in 'THE ASTRONOMICAL AND PHILOSOPHICAL BALANCE' OF LOTHARIO SARSI OF SIGUENZA* […], as translated by Stillman Drake, in Galileo (1957), p. 229. The word *assay* is related to *essay*, "a short piece of writing on a particular subject, an attempt or effort", both coming via Old French *essayer* "weighting" from Latin *exagium* "ascertain, weight".

referred to astrology and the ruder instrument used, among others, in market squares and in the greengroceries.[162]

Seen from this perspective, it is therefore somewhat surprising that after all, Galileo was not able to free himself from the traditional belief which considered the circular form as the only possible form in astronomy. According to the art historian Erwin Panofsky (1892–1968), Galileo kept his firm opinion on the superiority of the circular orbits not only for mathematical and mechanical reasons but also because of the aesthetic prejudices of his times.[163] Panofsky mentions that unlike the "perfect" circle, the "distorted" ellipse was never seen in High Renaissance art but was cherished in Mannerism, which loved "distorted" figures in general.[164] Panofsky quotes Galileo: "only circular motion is naturally appropriate to the bodies constituting the universe and disposed in the best order; rectilinear motion has been assigned by nature to the bodies and their parts whenever they are disposed in bad order, outside their proper places."[165] Obviously, Galileo also felt that elliptic, and not just rectilinear motion, was a sign that the planets were out of their "proper places". In this question concerning the superiority of the circular form in the celestial realms, even Galileo was, like so many of his contemporaries, still bound by the Aristotelian spirit.[166]

Galileo was certainly right when he said: "for building walls a square shape is more perfect than the circular, and for wagon wheels the circle is more perfect than the triangle", meaning that different shapes have different "functional" properties which we humans have learned to utilize in our daily life. In nature, different forces and processes sometimes also produce such simple geometric shapes, which are independent of any aesthetic or cultural prejudices. But whence in nature should we look for such precise geometric forms?

[162] Stillman Drake, *Galileo at Work; His Scientific Biography*, 1978, p. 284. The dispute continued, and later, Grassi referred to *Il Saggiatore* and Galileo as *assagiatore*, meaning "wine-taster" in Italian. See Drake (1957), p. 227, note 11.

[163] Erwin Panofsky, *Galileo as a Critic of the Arts*, 1954, p. 31. Panofsky's study was published in an abridged form in *Isis*, Vol. 47, No. 1 (March 1956), pp. 3–15 plus 9 pages of images. The treatise is also available online, for example, at http://www.jstor.org/stable/227542.

[164] *Ibid*. p. 25. Panofsky continues: "In painting it [the ellipse] does not occur until Correggio […]; in sculpture, not until Pierino da Vinci and Guglielmo della Porta; in architecture – apart from Michelangelo's first project for the Tomb of Julius II […] where it crept in, as it were, as an interior feature, invisible from without – not until Baldessare Peruzzi."

[165] Panofsky (1954), p. 25.

[166] Samuel I. Mintz, "Galileo, Hobbes, and the Circle of Perfection", *Isis*, Vol. 43, No. 2 (July 1952), pp. 98-100.

According to Greek historian Herodotus (*c.* 484–425 BC), geometry arose as a practical land-surveying skill in ancient Egypt, where it was used to (re-) establish the boundaries of the arable fields after the fertile floods of the Nile.[167] This practical earth-measuring activity also gave the more abstract geometry its current name in the Greek language: *geo* (earth, land) + *metron* (measure) as the Greeks started first to study geometry for its own sake, without immediate practical aims. They deduced that geometric truths are absolute, eternal facts, independent of time, place and human actions.

Even the Universe seemed to reflect simple geometry in its constitution, as the heavens with their myriads of stars give the *impression* of revolving around the Earth once per day as if fixed to a huge celestial "sphere". Greek scholars also understood that Earth itself was spherical like the Sun and the Moon.[168] In addition to visible forms, some philosophers concluded that in geometry there is more than meets the eye. Plato, for example, wrote in *Timaeus* (*c.* 360 BC)[169] that all the atoms of the four classical "elements" – earth, air, water, and fire – were strictly geometrical and regular in shape even if we are not able to see these particles and their supposed symmetric forms.[170]

All five possible regular convex solids, that is, the tetrahedron, hexahedron (cube), octahedron, dodecahedron, and icosahedron, were first mathematically described by Theaetetus of Athens (*c.* 417–369 BC), a contemporary of Plato, after whom these five *Platonic solids* were eventually

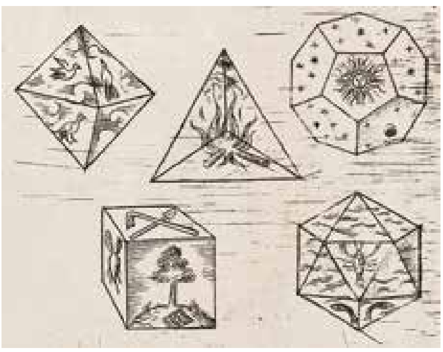


Figure 3.3 The five Platonic solids and the corresponding elements depicted in *Harmonices Mundi* (1619) by Johannes Kepler, a detail of plate IV; see Fig. 6.3 for the complete image.

167 Herodotus, *The Histories*, A. D. Godley (ed.), 1920, book 2, chapter 109, also available online at http://www.perseus.tufts.edu/hopper/text?doc=Perseus%3atext%3a1999.01.0126 (accessed 2016-07-05).

168 Eratosthenes of Cyrene (c. 276–c. 195 BC) even calculated Earth's circumference with a surprisingly correct result using only his reasoning and a few simple known facts.

169 http://oll.libertyfund.org/titles/plato-dialogues-vol-3-republic-timaeus-critias (accessed 2015-08-21).

170 Some scholars have speculated on whether Plato also implicitly hypothesized the existence of irregular atoms or particles in addition to the five regular ones. This view was discussed and opposed, for example, by Gregory Vlastos in his essay "Plato's Supposed Theory of Irregular Atomic Figures", *Isis*, Vol. 58, No. 2 (Summer 1967), pp. 204–209.

named.[171] In *Timaeus*, Plato associated earth with the stackable cube, air with the "mobile" octahedron, water with the "round" icosahedron[172], and fire with the "sharp-cornered" tetrahedron.[173] According to Plato, the fifth solid, the dodecahedron, "God used in the delineation of the universe."[174] By this he probably meant that the twelve faces of the dodecahedron somehow correspond with the twelve signs of the zodiac.

Kepler and the Platonic Solids

Johannes Kepler was almost obsessed with the Platonic solids in his first major astronomical work *Mysterium Cosmographicum*, printed in 1596 in Tübingen.[175] Unlike Plato, who associated the regular solids with infinitesimally small particles, Kepler associated the Platonic solids with the vast scales of planetary orbits. Kepler was convinced he had revealed the geometric plan according to which God had planned the Universe, and his *Mysterium Cosmographicum* describes how this cosmographic plan is based on the five Platonic solids. While teaching mathematics and astronomy at the protestant school in Graz, Kepler had an epiphany, either on July 9th or 19th, 1595, as Kepler himself quite precisely informs us.[176] He understood that as every regular polygon defines a proportion of two circles, one inscribed and one circumscribed, the relative proportions of these circles could

[171] See, for example, Sir Thomas Heath, *A History of Greek Mathematics*, Vol. 1, 1921, pp.159–162, also available online at https://archive.org/details/cu31924008704219 (accessed 2015-08-14). Theaetetus was also the main character in two dialogues of Plato: in the *Sophist* and, *mirabile dictu*, in the *Theaetetus*.

[172] Strange as it may sound, studies have recently shown that sometimes under normal conditions, water molecules (H_2O) can actually connect to form relatively stable icosahedral clusters. See, for example, O. Loboda and V. Goncharuk, Theoretical study on icosahedral water clusters, *Chemical Physics Letters*, Vol. 484, Issue 4–6 (Jan. 2010), pp. 144–147, or see http://www1.lsbu.ac.uk/water/icosahedral_water_clusters.html (accessed 2016-06-06), a site maintained since 2012 by Emeritus Professor Martin F. Chaplin, Fellow of the Royal Society of Chemistry. His site is a real cornucopia of water structures, listing, among others, over 2,500 references for published scientific articles related to different structures of water.

[173] Chapters 53–58 in Jowett's 1892 edition of Plato's *Timaeus*.

[174] *Ibid.* Chapter 55.

[175] The original *Mysterium Cosmographicum* 1596 is available in Latin scanned, for example, at http://dx.doi.org/10.3931/e-rara-445 (persistent link), and the second edition (1621) is available at https://archive.org/details/prodromusdissert00kepl (both accessed 2015-08-14). To my knowledge, the only published English translation so far of the whole book is the 1981 *Mysterium Cosmographicum – The Secret of the Universe*, translated by A. M. Duncan, introduction and commentary by E. J. Aiton, with a preface by I. B. Cohen.

[176] *Mysterium Cosmographicum* (1596), the story is in the preface on p. 8, and a diagram clarifying the idea n p. 9. Alternatively, see Judith Field, *Kepler's Geometrical Cosmology*, 1988, pp. 47–48, or Rhonda Martens, *Kepler's Philosophy and the New Astronomy*, 2000, pp. 12–13.

perhaps be matched with the proportions of the planetary orbits. Unfortunately, the proportions of the circles obtained from the two-dimensional regular polygons did not match with the celestial ones at all. Nevertheless, Kepler was adamant and tested his idea again with the three-dimensional regular polyhedrons, this time with better results.[177]

Kepler realized that the radii of the inscribed and circumscribed spheres defined by the three-dimensional Platonic solids could be combined to match with the radii of the orbits of the planets. Kepler's model was *heliocentric*, that is, it had the Sun – and not the Earth – as the centre of the universe, meaning that the Earth was thrown out of its assumed central position of the universe and demoted to a mere planet among other planets.[178]

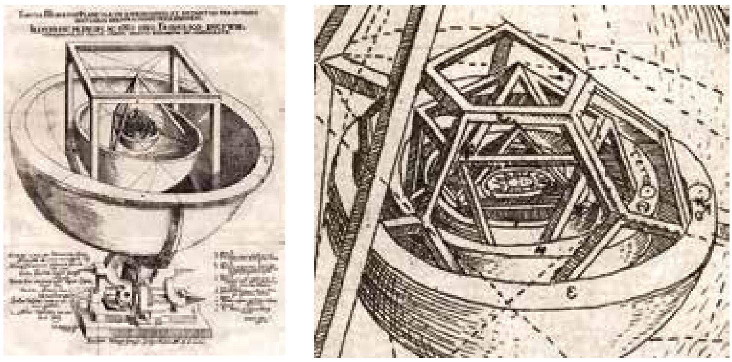


Figure 3.4 Kepler's model of the Universe shown in the *Mysterium Cosmographicum* (1596, after p. 24), the complete model (left), and a detail of its centre, showing the smaller orbits nearer the Sun (right).[179]

[177] Field (1988), pp. 45–62 and Martens (2000), pp. 39–48.

[178] Even after this, it still seemed that the Sun, and not the Earth, was the central body of the whole "universe". Giordano Bruno (1548–1600), who dared to propose that the Sun might be just one star among others, possibly with their own habitable planets, and who speculated that the universe itself might be infinite, was judged by the Inquisition and burnt at the stake at Campo de' Fiori in Rome.

[179] As so often happens, depicting regular polyhedrons here has also been a real challenge to the illustrator; see, for example, the distorted faces of the dodecahedron at right. On the other hand, considering that the illustrator in this case, Christophorus Leibfried, managed to draw all polyhedrons so well inside each other, the result can be seen as a success. The illustrator's name is visible in the print, which is dated 1597, not 1596, as in the book. The foldout illustration, which is larger than the pages of the book, was apparently printed and glued afterwards to the book, between pages 24 and 25. Leibfried is also mentioned by Gerhard Betsch in his essay "Kepler's Theory of Highly Symmetric Plane Figures and Solids, [Mysterium Cosmographicum 1596–1996]", *Acta Historiae Rerum Naturalium Necnon Technicarum*, New Series, Vol. 2 (1998), p. 116, Fig. 4.

Nicolaus Copernicus (1473–1543) had earlier proposed this heliocentric model, and Kepler's *Mysterium Cosmographicum* of 1596 was actually the first account of the Copernican theory, which emphasised its systematic aspects.[180] The traditional geocentric model had seven planets[181] orbiting Earth: the Moon, Mercury, Venus, the Sun, Mars, Jupiter and Saturn, but Kepler needed exactly six planets for his explanation to work. There exist only five regular polyhedrons, and if these are placed between the spheres of the planets, there must be exactly six planets. Thus, Kepler's theory "explained" not only the relative sizes of the orbits of the planets but also their number.

Kepler's model had the spheres of six planets – Mercury, Venus, Earth, Mars, Jupiter, and Saturn – placed around the Sun and held in correct distances from each other by gigantic invisible Platonic bodies: the octahedron, icosahedron, dodecahedron, tetrahedron and cube. The relative sizes of the planetary orbits proposed by this model were actually very accurate compared to the astronomical observations of his day.[182] Later, Kepler received permission to use the best observations of his time, made by the Danish astronomer Tycho Brahe (1546–1601). Working with this better data, Kepler realized first that the orbit of Mars and soon after that *all* the orbits of the planets were not circles but ellipses.[183] Eventually this discovery led him to formulate what is nowadays known as *Kepler's laws of planetary motion*, the first two of which were published in his *Astronomia Nova* (1609) and the third one in his *Harmonices Mundi* (1619).[184] Kepler's three

[180] Field (1988), p. 32.

[181] Ancient Babylonians used these seven planets, which in turn were named after gods, to number and name days in repeating handy but astronomically artificial units, that is, *weeks*. The Romans continued this system, and in many cultures, the weekdays are still named in this manner, for example, Saturn-day, Sun-day, Moon-day. In English, Nordic-Germanic mythology has replaced Latin names, for example, Thursday (Thor's day) instead of *Jovis dies* "Jupiter's day", in the days of the week. The word *planet* means "wanderer" in Greek, and the term referred to the fact that in the night sky, planets move, or "wander", compared to the "fixed" stars. The numbering of the spheres in medieval times often ran from the outermost Empyrean – the dwelling-place of God, angels, and the blessed – to the innermost Earth; thus the Earth was not only the "centre" but also the lowest part of the creation. As Hell was assumed to lie under the Earth, the medieval world-view was not only "geocentric", but also, as Arthur O. Lovejoy wrote: "diabolocentric". See Marjorie Hope Nicolson (1960), p. 28, or A. O. Lovejoy, *The Great Chain of Being: A Study of the History of an Idea*, 1948 (1936), pp. 101–102.

[182] Field (1988), chapter 3, pp. 30–72; see especially p. 38.

[183] See, for example, Curtis Wilson, "Kepler's Derivation of the Elliptical Path", *Isis*, Vol. 59, No. 1 (Spring, 1968), pp. 4–25, or Job Kozhamthadam, *The Discovery of Kepler's Laws; The Interaction of Science, Philosophy, and Religion*, 1994, especially chapter 9, pp. 199–245.

[184] Online versions are available at https://archive.org/details/Astronomianovaa00Kepl and http://dx.doi.org/10.3931/e-rara-558 (persistent link) for *Astronomia Nova,* and https://archive.org/details/ioanniskepplerih00kepl and http://dx.doi.org/10.3931/e-rara-8723 (persistent link) for *Harmonices Mundi* (all four sites accessed 2015-09-01).

laws are nowadays regarded as the cornerstone of modern astronomy and physics. The idea of a geometric physical model defined by nested Platonic solids was considered seriously again at least by one competent scientist in relatively recent times, albeit not in astronomy but surprisingly enough in nuclear physics.[185]

Ellipses provide an interesting example from the perspective of this thesis. They have been studied in geometry since ancient Greece, and from this point of view, they can be seen as *artefacts* of human culture. On the other hand, after Kepler›s discovery of ellipses in physical nature it would be incorrect to call ellipses «artefacts». Ellipses, and other conic sections as well, are examples of such artificial geometric objects, which were first constructed by the human intellect, but later turned out to have identical «non-artefactual» counterparts in nature. In the case of ellipses and other conic sections, it would take almost two millennia to discover their «nature-born» counterparts.

Kepler was proficient not only in astronomy but particularly in geometry and mathematics in general. Even his discovery of orbital ellipses did not make Kepler completely reject his idea of a universe based on Platonic solids or ideas of harmonies and regular structures in nature. Kepler studied not only actual (or hypothetical) regularities in nature. He also studied potential ways in which regular geometric structures *can* exist, which related directly to the study of *tilings* in the two-dimensional plane, and to the study of packing repeating objects in two or three dimensions.[186] I will return more closely to this topic in Chapter 6. Kepler was the first person among western scientists to pay closer attention to the theory of packing uniform spheres in three-dimensional space. These studies of packing spheres were inspired by his investigations of tiny snow crystals.

[185] In the 1980s, American physicist, chemist and engineer Robert James Moon (1911–1989) devised a model (the "Moon model") of the arrangements of protons and neutrons in the atomic nucleus based on nested Platonic solids. This model was directly related to Kepler's work published in *Harmonices Mundi*. See https://en.wikipedia.org/wiki/Robert_James_Moon or http://www.21stcenturysciencetech.com/moonsubpg.html (both 2016-06-06). As far as I can tell, the "Moon theory" is not seriously considered by any specialists in the field, and much more precise – and tested – models of the shape, vibration and other dynamic behaviour of the atomic nuclei have been available for more than half of a century. For some relatively new results, see, for example, L. P. Gaffney *et al*., "Studies of pear-shaped nuclei using accelerated radioactive beams", *Nature*, Vol. 497 (9 May 2013), pp. 199–204. I am grateful to co-author of the paper and member of the research team, physicist Janne Pakarinen, for bringing this 2013 article to my attention.

[186] Kepler's studies of tilings were published in his *Harmonices Mundi* (1619). In the Islamic world, various geometric tilings have been used and studied for centuries. Unfortunately, my knowledge of the Islamic tilings is rather limited, and the subject forms a complete world of its own. Hence, I decided not to incorporate Islamic tilings in this thesis.

The Six-cornered Ciphers

Snow crystals always appear to be of hexagonal symmetry, and Kepler seems to have been the first western scientist to point this out.[187] In 1555 in Rome, the Swede Olaus Magnus (1490–1557), the Roman Catholic Archbishop[188] of Uppsala, published his histories of the Northern people[189] containing images of frost and snow[190], although the Chinese philosopher Han Ying had already written of the six-pointed nature of the snow flakes around 135 BC. [191] Unlike the later works of Kepler and Descartes, the woodblock prints used in Olaus Magnus' book did not acknowledge the typical six-fold nature of snow crystals, notwithstanding one single hexangular snowflake in the image in question.[192]


Nam ſi ad ſtructuram ſolidorum quam poteſt fieri arctiſſimam progrediaris, ordinesq́; ordinibus ſuperponas, in plano prius coaptatos, aut ij erunt quadrati A aut trigonici: B ſi quadrati aut ſinguli globi ordinis ſuperioris ſingulis ſuperſtabunt ordinis inferioris aut contra ſinguli ordinis ſuperioris ſedebunt inter quaternos ordinis inferioris. Priori modo tangitur quilibet globus à quatuor circumſtantibus in eodem plano, ab

Figure 3.5 A quadrangular (A) and a triangular (B) packing of uniform spheres depicted in Johannes Kepler's *Strena Seu De Nive Sexangula* (1611).

[187] Philip Ball, *Nature's Patterns: a Tapestry in Three Parts; Branches*, 2009, p. 2, and Ukichiro Nakaya, *Snow Crystals: Natural and Artificial*, 1954, p. 1. Ukichiro Nakaya (1900–1962) was the first person to succeed in producing artificial snowflakes in the laboratory. His book is a classic in the scientific study of snow crystals. His daughter Fujiko Nakaya (b. 1933) is an artist famous for her fog sculptures, creating the world's first in 1970.

[188] As Sweden was no longer Roman Catholic, this title was only nominal.

[189] In Finnish, there is a great book available about snow crystals and their study throughout western history by Raimo Lehti (1931–2008), *Lumihiutaleet ja maailmankuvat*, [snowflakes and worldviews] 2000 (1998).

[190] Olaus Magnus, *Historia de Gentibus Septentrionalibus*, Book 1, chapter 22. The image of snow and frost is on page 37 in the first 1555 edition, available online, for example, at https://archive.org/details/bub_gb_O9lEAAAAcAAJ (accessed 2015-08-19).

[191] Ball, *Branches* (2009), p. 2.

[192] *Ibid*. On the next page (p. 38), Olaus shows one classic way of enjoying fresh snow: boys having a snowball fight with a mighty snow castle. *Sunt pueri pueri et pueri*…

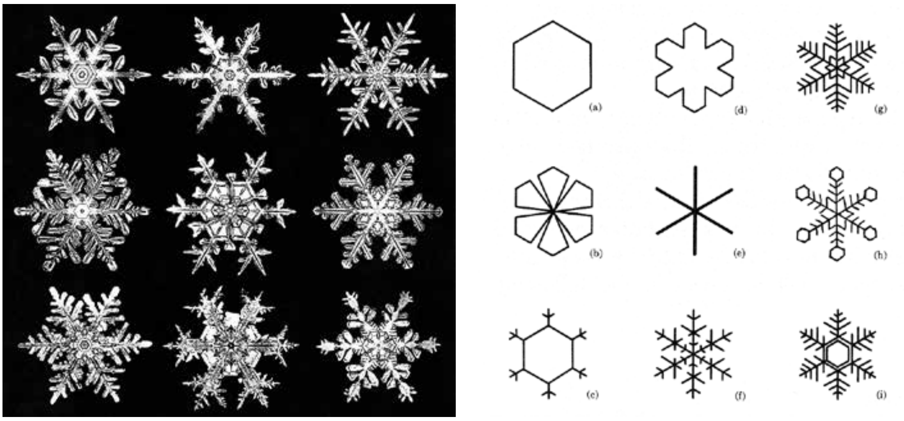


Figure 3.6 Wilson A. Bentley, photographs of snow crystals, 1931, detail (left), and Ukichiro Nakaya, the classification of regular planar snow crystals, 1954, detail (right).

In the beginning of 1611, in between his larger scientific works, Kepler published the short, 24-page study *Strena Seu De Nive Sexangula* as a present dedicated to his friend and benefactor baron Wackher von Wackenfels.[193] The title of the book means "A New Year's Gift or On Hexagonal Snow".[194] Despite the name of Kepler's book and the fact that it contains three images, none of these images actually depicts a snow crystal.[195] What Kepler was looking for was not so much about describing real physical snowflakes[196] but to ponder the processes and principles

[193] The original 1611 edition is very rare, but it is available scanned, for example, at https://archive.org/details/den-kbd-pil-21055000404F-001 (accessed 2015-08-19). A critical edition of it is Johannes Kepler, *The Six-Cornered Snowflake*, Oxford University Press 1966, which is a bilingual Latin-English edition. A new printing was completed in 2014. This reprinting was clearly a response to another Latin-English bilingual edition published by Paul Dry Books, Philadelphia in 2010, which is based on the Latin text edited by M. Caspar and F. Hammer in Johannes Kepler *Gesammelte Werke*, München 1941, Vol. IV, pp. 259–280. From other German translations of this snowflake-book, I will mention *Neujahrsgabe oder vom sechseckigen Schnee*, M. Caspar (ed.), Berlin 1943, *Vom sechseckigen Schnee*, Dorothea Goetz (ed.), Geest & Portig, Leipzig, (DDR) 1987, and *Vom sechseckigen Schnee*, Lothar Dunsch (ed.), Hellerau-Verlag 2005. See also Cecil J. Schneer [not *Schnee*], "Kepler's New Year's Gift of a Snowflake", *Isis*, Vol. 51, Part 4 (December 1960), pp. 531–545.

[194] Kepler's small tractate is sometimes referred to in literature briefly as *Strena*, which is the "New Year's Gift" part of its title, or as *De Nive Sexangula*, which is the "On Hexagonal Snow" part of its title. Between them, *seu* means "or" in Latin.

[195] See, for example, https://archive.org/details/den-kbd-pil-21055000404F-001 (accessed 2015-08-19)

[196] As was the American farmer and enthusiastic amateur photographer Wilson A. Bentley

behind the mysterious and so far unexplained six-fold symmetry-principle, which snowflakes so rigidly seem to obey. But where does this six-fold symmetry come from?

In his *De Nive Sexangula*, Kepler hypothesized that the six-cornered external structure of snowflakes reflected their internal structure. He speculated that the internal structure of the snow crystals could be depicted with uniform regular spheres representing atoms or other corresponding unobservable tiny particles. If a fairly large number of uniform spheres or discs are placed on a flat plane, as close to each other as possible, the result is a regularly repeating structure. In such a triangular structure, the centres of the spheres, or discs, take positions in the vertices of equiangular triangles, and six spheres always surround one at the centre; see Fig. 3.5 (B). Two decades after Kepler, the French philosopher René Descartes (1596–1650) also made observations on snow crystals. His drawings of them were published in 1635 in Amsterdam, and they are nowadays thought to be the first actual realistic-looking depictions of the six-folded snowflakes.[197] Descartes cherished the same idea as Kepler, as evidenced in his writing: "And the small little clusters of ice [...] are obliged to arrange themselves in such a way that each has six others surrounding it; one cannot conceive of any reason that would prevent them from doing this, because all round and equal bodies that are moved in the same plane by the same kind of force naturally arrange themselves in this manner, as one can see by an experiment, in throwing a row or two of completely round unstrung pearls confusedly on a plate, and shaking them, or only blowing against them slightly, so that they approach one another."[198]

The English chemist and father of modern atom theory, John Dalton (1766–1844), almost two hundred years later published the same explanation as Kepler and Descartes for the shape of a snowflake in his *A New System of Chemical Philosophy* (1808). He explained the images in his book's Plate 3, reproduced here as Fig. 3.7, in the following way: "Fig. 5 represents one of the small spiculæ of ice

(1865–1931), who captured more than 5000 microscopic images of snowflakes in his lifetime. *Snow Crystals* by W. A. Bentley and W. J. Humphreys, 1931, a classic of the subject, is still in print to this day by the Dover Publications from 1962 onwards. To achieve greater visual impact and sharper edges, Bentley used to render the backgrounds of his white snowflakes totally black by cutting them out carefully with scissors from the duplicates, but not from the original negatives, as it is sometimes stated. See Bentley (1931), p. 14 and Fig. 3.6 in this thesis.

197 Nakaya (1954), p. 1.

198 John G. Burke, *Origins of the Science of Crystals*, 1966, pp. 35–38. Burke refers in turn to Charles Adam and Paul Tannery (eds.), Œuvres de Descartes, 12 volumes, Paris, 1897–1913, Vol. VI, pp. 288.

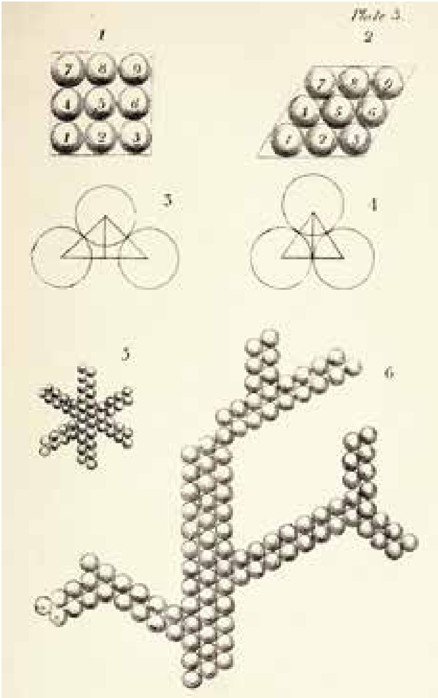


Figure 3.7 Plate 3 with its six figures from John Dalton's *A New System of Chemical Philosophy* (1808).

formed upon the sudden congelation of small water cooled below the freezing point […]. Fig 6 represents the shoots or ramifications of ice at the commencement of congelation. The angles are 60° and 120°."[199]

It should be noted that the small curvy areas in between the spheres, or discs, are always left uncovered as a consequence of the circular form. With regular uniform triangles, squares, or hexagons, 100% of the plane can be covered. [200] With uniform spheres, or discs, the densest possible packing covers only *c*. 90.69% of the plane.[201] This optimal configuration was depicted by Kepler; Fig. 3.5 (B), and by John Dalton; Fig. 3.7 (2). The square-based packing of spheres, or discs, is also seen in Figs. 3.5 (A), and 4.7 (1). This quadrangular packing is even less dense, covering only *c*. 78,54% of the plane.

Emergence, Self-Similarity, and my Paintings

In my own paintings, I have often made direct references to snowflakes. They have represented properties I am much fascinated by: *emergence* and *self-similarity*. Emergence refers to some notable qualitative change which cannot be foreseen from the properties of the subject before the change takes place.[202] In the case of the snowflakes, this unexpected qualitative change is the six-fold structure, which

[199] Burke (1966), p. 121, Fig. 15, or see Dalton's original text and Plate 3 (pp. 218–219), for example, at https://archive.org/details/newsystemofchemi01daltuoft (accessed 2016-02-15).

[200] Grünbaum and Shephard (1987), pp. 58–64.

[201] Or $\pi / \sqrt{12}$ to be exact. See, for example, Aste and Weaire (2000), p. 14.

[202] The most amazing examples of emergence are the birth of life and later the development of consciousness. Both are truly remarkable qualitative changes in "dead" matter.

is not present in liquid water, at least not in a perceivable manner. Self-similarity refers to a structure in which details are miniature copies of the whole. In the case of the snowflakes, the smaller branches look more or less the same as the largest six branches.

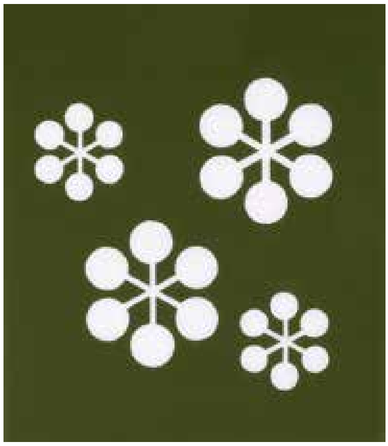


Figure 3.8 Markus Rissanen, [M84] *Cool Composition IV*, acrylic and epoxy resin on MDF, 28x24cm, 2004.

Snowflakes also appear to "self-organize". On the surface, it looks as if the six branches of the snowflake somehow communicate or at least work in astonishing unison in their complex-looking development. However, most of these properties are rather illusory in the case of snowflakes. The six branches of a snowflake do not in fact communicate with each other. Their near-perfect similarity is only a consequence of the similarity in the micro-scale climatic conditions regulating the growth of the snow crystal.[203] Even the apparently self-similar structure in real snowflakes does not actually continue very deep: after the main branches, there are only one or two levels of secondary twigs (or sub-branches) present.[204] A real snowflake is thus far from the ideal self-similar structure, which maintains its self-similarity theoretically *ad infinitum*. Often the actual forms of the crystal's edges are not geometrically straight and angular but round and curvy due to natural forming processes, sublimation or melting.[205]

I have certainly felt fondness for snowflakes for many years while doing my paintings not least for the honourable role they have played in the history of studying the forms and structures of nature. So far, I have not tried to imitate their true appearances, representing them only in a very much-simplified form, like much of the other subject matter in my works. Snow crystals might not fulfil the strictest scientific criteria for being the most valid examples of emergence, self-similarity, or self-organization. Yet, fascinated by their characters, I have used snowflakes as simple symbols for complex matters.

203 Nakaya (1954), *passim*.

204 *Ibid*., pp. 78–88: *General Classification of Snow Crystals*.

205 *Ibid*. See, for example, plates 23/105 (p. 339), 54/284 (p. 370), and 85/468 (p. 401).

4 | The Forms of Nature

In this chapter, I discuss how simple forms, such as the triangle, square, circle, and tree-like form are found in physical nature. I present how animals and physical forces are sometimes behind the formation of such simple shapes. For the sake of completeness, some *spiral-forms* in nature are also mentioned. The tree-like form leads the discussion into two even more dynamic systems: *fractals* and *organic forms*. In the first footnote, I list some books dealing with the very broad theme of the "forms of nature".

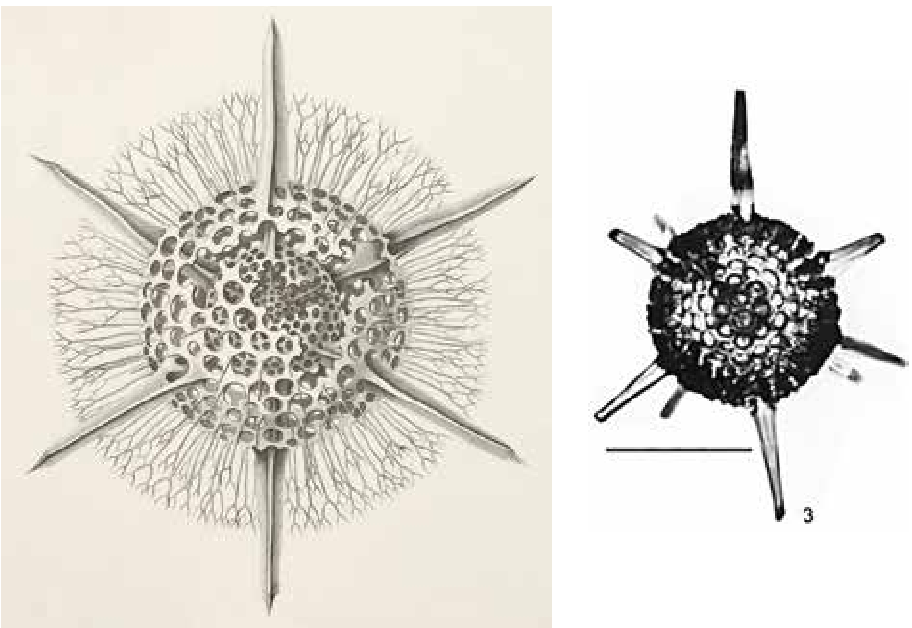


Figure 4.1 Radiolarian *Actinomma drymodes* from Ernst Haeckel's 1862 monograph *Die Radiolarien* (left), and a modern microscope photograph of radiolarian *Actinomma delicatulum* (right). I was unable to find "Actinomma drymodes" from the modern nomenclature of radiolarian, but I believe these images represent the same species.

Forms, patterns, and structures present in nature must have fascinated humans as long as our species has had some minimum capacity of schematization. The habit of observing and contemplating the development and character or "design" of these forms must have its roots deep in our cognitive prehistory. Throughout history, the forms of nature have fascinated innumerable humans quite independent of whether they have held a theistic or an atheistic worldview. I will next examine some forms found in nature which are the same as those introduced as basic shapes in the visual arts, namely the triangle, the circle, and the square. In addition, I will write of spirals and tree-like forms, which some may consider, perhaps even more than the angular shapes, as basic forms of nature. The tree-like shape works also as a bridge to fractals, which constitute a large class of forms found in nature but not in classic geometry. There exists, naturally, a wide literature[206] on the subject, extending from scientific monographs via more general philosophical treatises and artistic analyses to recreational photographic books of, for example, colourful flowers.

[206] There is almost a literary genre on the "forms of nature". A classic is *On Growth and Form,* 1917, by the Scottish mathematical biologist D'Arcy Wentworth Thompson (1860–1948). The revised second edition in 1942 contains 1116 pages. The book has been in print ever since. Thompson emphasized the structural and mechanical "engineering" explanations but partly ignores the role of the theory of evolution in the development of plant and animal forms. The English science writer Philip Ball has described the book's current position in the following way: "Like Newton's *Principia,* D'Arcy Thompson's *On Growth and Form* is a book more often name-checked than read." Ball, (2013). Another colossal, slightly older book is J. Bell Pettigrew's *Design in Nature,* 1908. Pettigrew's three-volume book is a cornucopia of images, and the broadness of its spectrum is well presented in its full title: *Design in Nature; Illustrated by Spiral and other Arrangements in the Inorganic and Organic Kingdoms as exemplified in Matter, Force, Life, Growth, Rhythms, etc., especially in Crystals, Plants, and Animals. With Examples selected from the Reproductive, Alimentary, Respiratory, Circulatory, Nervous, Muscular, Osseous, Locomotory, and other Systems of Animals.* From the Scottish financier and philosopher of science, Lancelot Law Whyte (1896–1972), two books about the philosophy of form in science and nature are worth mentioning: *Aspects of Form; A Symposium on Form in Nature and Art,* 1951, and *Accent on Form; An Anticipation of the Science of Tomorrow,* 1955. The former has an interesting 'Chronological Survey on Form' from 25,000 BC to 1942. Photography has provided a very suitable medium for recording the forms of nature, and *Urformen Der Kunst: Photographische Pflanzenbilder,* 1929, by Karl Blossfeld (1865–1932) is a classic example of such work. The American photographer Andreas Feininger (1906–1999) produced numerous books since the 1960s depicting the forms of nature, including publications such as *Forms of Nature and Life* (1966), *The Anatomy of Nature* (1979), and *In a Grain of Sand; Exploring Design by Nature* (1986). Another category related to depicting forms of nature is books in which the works of science are compared to those of art. A classic among such studies is Gyorgi Kepes (ed.), *The New Landscape in Art and Science,* 1956. Another example is *Behind Appearance: A Study of the Relations between Painting and the Natural Sciences in This Century,* 1970, by the British biologist and philosopher C. H. Waddington (1905–1975). Another such book is Georg Schmidt and Robert Schenk (eds.), *Kunst und Naturform – Form in Art and Nature – Art et Nature,* 1960. I will mention also works by Peter S. Stevens, *Patterns in Nature,* 1976, and by Peter Pearce, *Structure in Nature is a Strategy for Design,* 1978. The former emphasizes more the visible shapes

Radiolaria

It is not always easy to define geometric forms, regularities and symmetries in visible nature, especially when it comes to living organisms. The German biologist, naturalist, physician and philosopher Ernst Haeckel (1834–1919) studied, among other things, small marine organisms, *radiolarians*, measuring typically 0,1–0,2mm. Haeckel was a radiolarian specialist who constructed their first comprehensive taxonomy.[207] He reported approximately 4,300 species and published several scientific articles and two scholarly monographs on radiolaria.[208] Radiolarians were included also in his most famous and splendidly illustrated book *Kunstformen der Natur* (1899–1904), published in English as *Art Forms in Nature*.[209] The exactitude and sincerity of Haeckel's depictions has been much discussed and even heavily criticised, especially when it comes to his illustrations of the development of embryos.[210] Philip C. Ritterbush, a historian of biology, wrote of Haeckel's

of living things, while the latter focuses more on the structures suitable for engineering. The Universal Node system (U.S. Patent 3600825) invented by Pearce and presented in his book was clearly a direct antecedent to the contemporary Zometool® geometric modeling kit (U.S. Patent 4701131). An example of newer books in the "forms of nature" genre is the 2009 trilogy *Shapes / Flow / Branches: Nature's Patterns: a Tapestry in Three Parts* by the aforementioned Philip Ball. It is most unfortunate that so many gray-scale images in this trilogy, at least in its original 2009 printing, were reproduced in a rather limited spectrum of middle-gray tones with a lack of pure black and white shades. The result was difficulties in seeing some details and a dull overall appearance. The trilogy was an expansion of Ball's earlier book *The Self-made Tapestry: Pattern Formation in Nature,* 1999, as it was also an apparent antecedent to his *Patterns in Nature: Why the Natural World Looks the Way It Does,* published in 2016.

207 Yoshihiro Tanimura and Yoshiaki Aita (eds.), *Joint Haeckel and Ehrenberg Project: Reexamination of the Haeckel and Ehrenberg Microfossil Collections as a Historical and Scientific Legacy*, Tokyo: National Museum of Nature and Science, 2009, available online at https://www.kahaku.go.jp/research/db/botany/ehrenberg/ehrenberg_top.html (2016-07-16).

208 *Die Radiolarien (Rhizopoda radiaria): eine Monographie*, Berlin: Georg Reimer, 1862, available online at https://archive.org/details/dieradiolarienrh00haec and *Report on the Radiolaria collected by H.M.S. Challenger during the years 1873-76*, Edinburgh, 1887, available online at https://archive.org/details/reportonradiolar00haecrich (both accessed 2016-07-15). See also Tuula Närhinen, *Kuvatiede ja luonnontaide*, (visual science and natural art) 2016, p. 135–145, (in Finnish), also available online at http://urn.fi/URN:ISBN:978-952-7131-16-9

209 Illustrations were first published as sets of ten separate sheets in 1899–1904, and as a two-volume book in 1904. Haeckel's book was a great influence on many artists and designers of the Art Nouveau / Jugend style in Belgium, France and Germany. A certain culmination was reached when the French architect René Binet (1866–1911) designed the main entrance portal of the *Exposition Universelle* of 1900 in Paris after a radiolarian à la Haeckel. See, for example, Lynn Gamwell (2002), p. 77, or Philip Ball, *Shapes,* (2009), pp. 33–49.

210 See, for example, Nick Hopwood, "Pictures of Evolution and Charges of Fraud: Ernst Haeckel's Embryological Illustrations", *Isis*, Vol. 97, No. 2 (June 2006), pp. 260–301.

illustrations: "Under the influence of his aspirations to find strict symmetry, Haeckel altered his drawings to conform to his belief in the geometrical character of organic form. A process of generalizing abstraction resulted in representations that were improvements upon nature. The observer who inspects a radiolarian under the microscope today will be disappointed at his impressions of reality as compared to the crisp and symmetrical outlines of Haeckel's superb lithographs."[211]

I am sure there are also many more depictions of nature in which the correspondence with "reality" is under suspicion, but the image in Fig. 4.1 (left) is one which was specifically used by Ritterbush in 1968, and the image in Fig. 4.1 (right), on theother hand, is a modern microscope photograph published at about the same time, in 1966.[212] I hope this comparison illustrates how difficult it is to draw the line between slavish depictions or photographs, and idealized or "improved" representations of forms in nature, which have, or seem to have, some more or less regular geometric character.

Triangle "the Simple"

The equilateral triangle is the simplest possible regular polygon in Euclidean geometry, and the triangle in general is the simplest two-dimensional closed, convex form that can be defined with a minimum number of straight lines.[213] If we start dividing an arbitrary two-dimensional figure with straight lines, we easily end up with a mesh of triangles. The triangle is the simplest rigid form made of straight lines; therefore, it is extensively used in engineering by nature and humans.

[211] Philip C. Ritterbush, *The Art of Organic Forms*, 1968, p. 64.

[212] The right-hand image is taken from http://www.radiolaria.org (accessed 2016-07-16), and the source is given as Richard N. Benson, *Recent Radiolaria from the Gulf of California*, PhD thesis, University of Minnesota, 1966. Another photograph in the site (Bjørklund & Benson 2003) also shows tiny fragile "hairs", or by-spines, or supplementary *spicules*, depicted by Haeckel, which are missing from this particular Benson 1966 photograph. The site also verbally describes the structure: "Test consisting of three concentric lattice shells joined by numerous […] three-bladed to cylindrical radial beams […] of nearly equal length. […] in one specimen a thin, delicate, spherical outer veil developed between distal branches of the by-spines. […] Slender supplementary spicules arising from wall of shell at points of junction of margins of pores; these spicules often broken off, even in specimens from plankton."

[213] A one-dimensional equivalent for the triangle is the straight line connecting its two end points, and the three-dimensional equivalent for the triangle is the tetrahedron, which is limited by four triangles enclosing its volume. In geometry, we can easily generalize the three-dimensional tetrahedron further to *n*-dimensions as a collection of *n*+1 points, which are all connected to every other point by straight lines. Such an object is called a *simplex*.

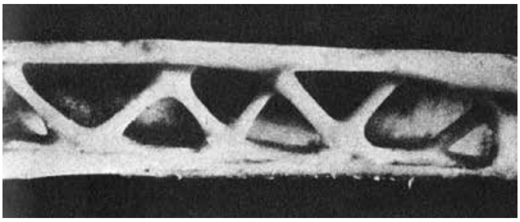


Figure 4.2 The metacarpal bone of a vulture, perfectly resembling the Warren truss, an engineering structure invented in 1848, from Oskar Prochnow, *Formenkunst der Natur* (1934).

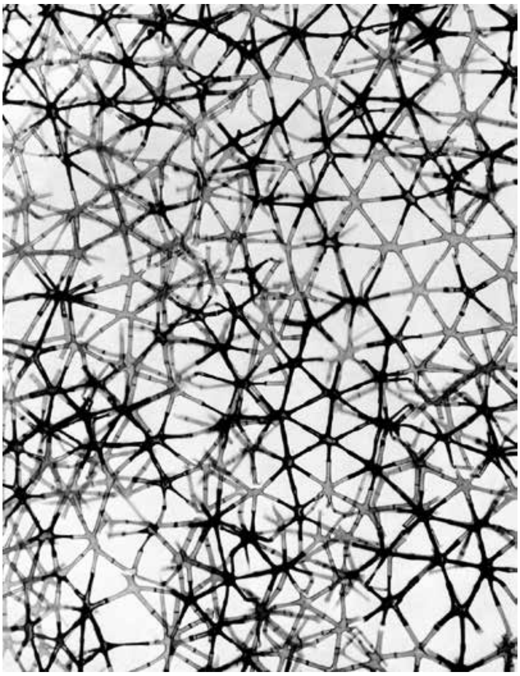


Figure 4.3 A microscopic photograph of the cell-beams in the stalk of a bulrush, from Carl Strüwe, *Formen des Mikrokosmos; Gestalt und Gestaltung einer Bilderwelt* (1955).

It is not so typical to see sharp triangles in nature with the unaided eye. We can see them occasionally in seashells or in the cracks of certain hard materials. With the aid of a light or scanning electronic microscope, which has a much better magnifying power than the traditional microscope using light, amazing triangular or rectangular shapes and structures can be seen. The tiny unicellular marine organisms *coccolithophores* and *diatoms*, measuring only some micrometers (10^{-6}m), often have a geometric shape.[214] The actual living cell of diatoms is surrounded by

[214] See, for example, F. E. Round, R. M. Crawford, and D. G. Mann, *The Diatoms: Biology & Morphology of the Genera*, Cambridge: Cambridge University Press, 2000 (1990) for diatoms and the site http://ina.tmsoc.org/Nannotax3/index.html (2016-06-22) for coccolithophores.

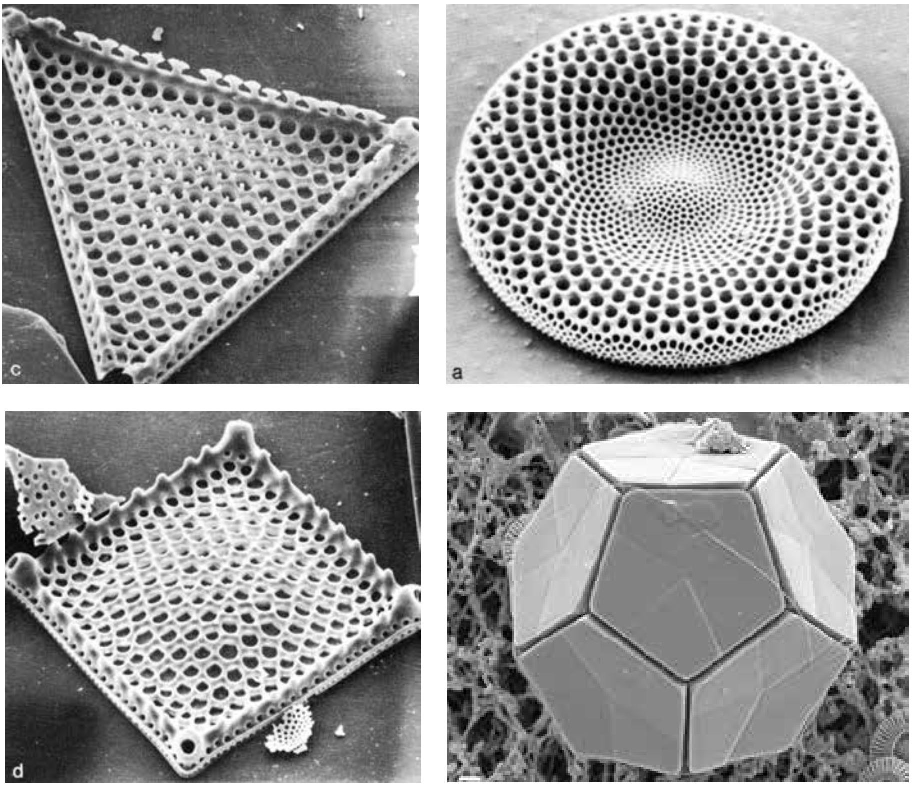


Figure 4.4 Three examples of diatoms with the characteristic overall shapes of a triangle, a circle, and a square. The triangle (c) and the square (d) are *Triceratium favus* diatoms, and the circular one (a) is the diatom *Craspedodiscus elegans*. The fourth image depicts an enigmatic and perfectly dodecahedral coccolithophore *Braarudosphaera*.

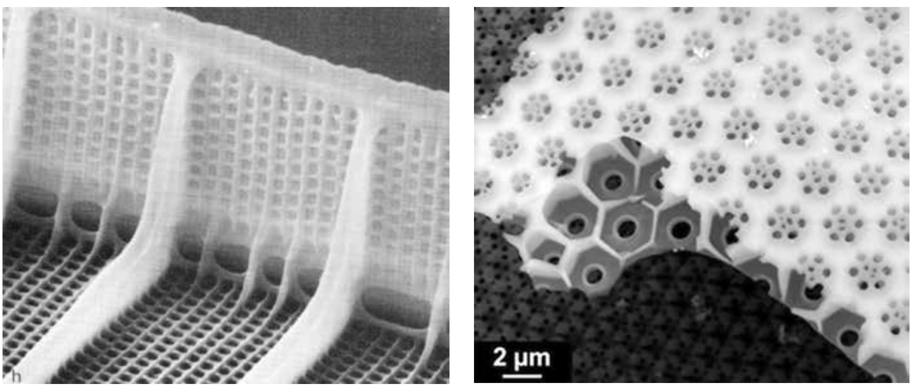


Figure 4.5 Natural engineering by diatoms made of hard glass-like silica: a structure resembling a typical angle iron in *Petrodictyon gemma* diatoms (left), and a magnification of the double-layered silica shell surrounding the actual living cell (right). All this is made by unicellular alga: a plant with no brain, eyes, or hands. Compare the scale of two micrometres (2 µm) with the man-made hi-tech structure seen in Fig. 5.6.

Figure 4.6 Naturally grown crystals of: (1) pyrite (upper left), (2) galena (upper right), (3) "smoky" quartz (lower right), and (4) another sample of pyrite (lower left). Note that the pentagonal faces of pyrite in the lower left image are *not* regular pentagons, and thus the dodecahedrons are *not* regular dodecahedrons.

frustule, a hard but porous translucent silica shell, often with regularly positioned minuscule and elaborate openings in it; see Fig. 4.5 (right).

In even smaller scale, the molecules and very atoms of solid matter often tend to organize themselves in repeating, simple geometric structures: "If we reduce the motion of a liquid, the links between molecules will become more stable. The molecules will then cluster together to form what is macroscopically observed as a rigid body. They can assume a random disposition, but an ordered pattern is more likely because it corresponds to a lower energy state."[215] In favourable conditions, these arrangements of molecules give rise to crystals, which are large enough to be seen with the naked eye.

The Square and the Cube

The square is the next regular polygon after the equilateral triangle. Like perfect triangles, perfect squares are rarely seen in nature with unaided eyes. Even if the triangle by its nature is the most simple and rigid two-dimensional form, which can be defined with a minimal number of straight lines, it is by no means the most natural or even the simplest form seen from the *human* perspective.

[215] C. Giacovazzo (ed.), *Fundamentals of Crystallography*, Oxford: International Union of Crystallography and Oxford University Press, 1992, p. 1.

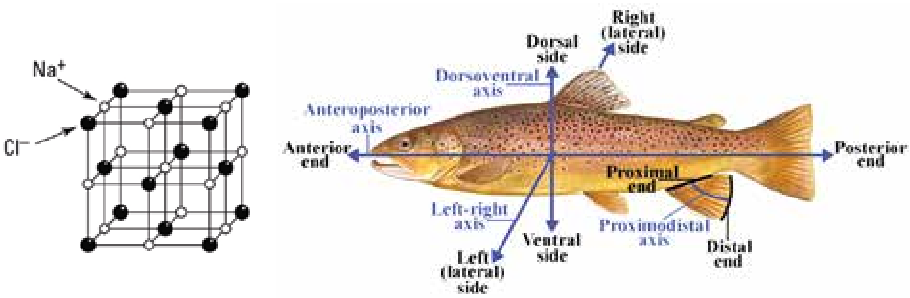


Figure 4.7 A cubic structure with its three orthogonal axes in the crystal of common table salt (left), and the three orthogonal axes which also define the anatomical directions in a vertebrate such as the fish (right).

Our western culture has a strong tendency to divide the space and directions around us in complementary pairs such as up–down, front–back and left–right. One can argue that this division of directions has its base in "natural" conditions. We humans live on the solid surface of the Earth, and it is the omnipresent gravity of our planet which defines the directions "up" and "down" as a given condition. This *vertical* up–down axis has as its perpendicular the *horizontal* plane with infinitely many "compass" directions in the horizontal plane. These two fundamental directions, vertical and horizontal, already define a simple rectangular structure of perpendicularly crossing lines, the simple result of the gravity of the planet we live on. The gravity of the Earth defines not only directions for us humans and other animals, but for all higher plants, such as the trees and the flowers, which have a strong tendency to grow in the direction "up". The limits of the effects of gravity on all living things on Earth seem to de defined either by the size of the organism or by the element it is living in.[216] Also, the bodies of all higher animals are functionally differentiated in directions called *anterior* and *posterior*, or unscientifically expressed as "head" and "tail", respectively. See Fig. 4.7 (fish).

The right-angled grid is the epitome of the order in human planning and construction: it can be applied from small bricks to a house, from the houses to a block, and from the blocks to a complete city.[217] Of course, houses and cities are

[216] In water, for example, the effects of gravity are much smaller compared to those on land and on air. It is therefore understandable why the animal and plant forms in the aqueous environment seem to know no limits in their surrealistic variety.

[217] The oldest cities planned in grid modules are found in India, Egypt, Greek and Rome. Many European cities have their origins in the Roman standardized military camp called *castrum*. See Sibyl Moholy-Nagy, *Matrix of Man: An Illustrated History of the Urban Environment*, 1968, 'The Modular Grid Plan', pp. 158–197, and Hannah B. Higgins, *The Grid Book*, 2009, 'Gridiron', pp. 49–77.

not produced by nature, but it is nevertheless interesting to compare these human-made structures to similar types of structures found in nature. As we are not used to encountering nature-made rectangular structures very often, it is understandable why such objects look so "unnatural", or as if they were designed by somebody with a purpose.

Sometimes the arrangements of, for example, microscopic cells of certain plants may also be very geometrically ordered. One example of such an order is the blue-green alga *Merismopedia*; see Fig. 4.8 below. Another rectangular blue-green alga is *Eucapsis*, closely resembling *Merismopedia*, except for the fact that instead of the cell division remaining on a plane, it occurs at successive right angles to one another in three dimensions.[218] The American biologist John Tyler Bonner (b. 1920) wrote of the regularity in such algae: "[T]he striking simple symmetry of the cells really has no parallel in the organic form. They are almost too geometrically simple and perfect to seem real and living."[219] Bonner continued his biological analysis concerning another elementary shape: "The sphere, especially the hollow sphere, is an extremely common shape assumed by a mass of dividing cells."[220]

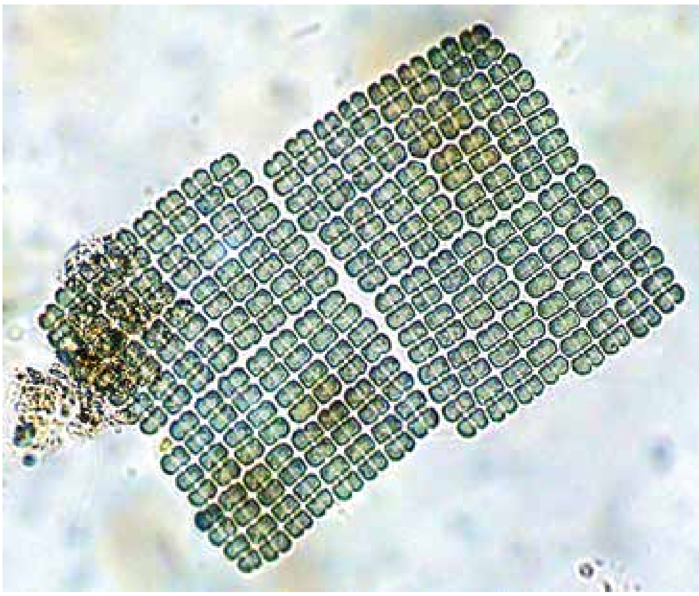


Figure 4.8 Cells of *Merismopedia* arranged in rectangular rows and columns on a plane.

[218] John Tyler Bonner, *Morphogenesis: An Essay on Development*, 1963 (1952), pp. 80–81.
[219] *Ibid*., p. 82.
[220] *Ibid*., p. 83.

The Circle and the Sphere: a Potato or Not a Potato?

The circle is one of those few geometric elementary forms which are perceivable for us humans in nature with our unaided eyes: The Moon and the Sun are circular forms we can perceive easily, as are also the circles and bubbles in water and froth. Some, albeit few, animals produce strikingly regular geometric patterns, the hexagonal honeycomb of the bees being surely one of the best-known examples of such structures. On some occasions, the purpose or the maker of the pattern is not immediately so obvious.

One such case is the circular and radially beautifully repetitive ephemeral patterns made by the puffer fish (*Torquigener* Tetraodontidae), which lives in tropic estuaries, see Fig. 4.9 above.[221] Some puffer fish males have the habit of producing this pattern (diameter *c*. 2 meters) in the sandy bottom by forceful movements of their fins. Circles do not last for long in shallow waters, as waves will smooth them out in few days. The fish itself is also visible in the image: swimming from right to left, almost at the very centre. The purpose of these "sand castles", perhaps unsurprisingly, is to catch the attention of female puffer fishes for mating.

The circle and the sphere are mathematically defined as the collection of points whose distance is constant from some fixed point, that is, the centre, either on a two-dimensional plane or in three-dimensional space, respectively.[222] The circle is the most symmetric form possible in two dimensions, just as the sphere is the most symmetric form possible in three dimensions. Packing round things in two-, three- or in higher dimensions is very much about studying the geometry of spheres. A sphere is often the optimal solution for many theoretical problems. For example, with some fixed surface area, the sphere has the maximal volume among all three-dimensional forms, and the circle has the largest area of all two-dimensional forms with a fixed perimeter.

The equilibrium of forces often produces round forms. Examples of this are the soap bubbles and their celestial cousins: stars and planets. All solid objects with a diameter larger than approximately 600 km are, by necessity, spherical due to the gravity of their own mass. No known material has enough compressive strength

[221] See http://www.bbc.co.uk/guides/zsjfyrd, or read the original scientific article by Hiroshi Kawase, Yoji Okata, and Kimiaki Ito, "Role of Huge Geometric Circular Structures in the Reproduction of a Marine Pufferfish", in *Scientific Reports* 3 (2013), article no. 2106; available online at http://www.nature.com/articles/srep02106 (both accessed 2016-08-06).

[222] In more advanced mathematics the spherical object can easily be generalized also into higher dimensions.

Figure 4.9 A circular pattern made in the bottom of the sea by a puffer fish.

Figure 4.10 A perfectly circular form seen once a month in the sky: the full Moon.

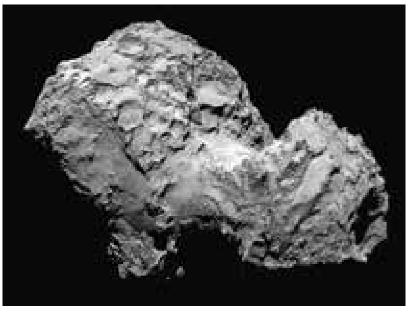


Figure 4.11 A relatively small and not-so-round comet 67P/Churyumov–Gerasimenko with a very rugged terrain has its maximum length and width measuring approximately 4 x 4 km, giving a mass which is not sufficiently large to force the object into a spherical shape. The spacecraft Rosetta's probe Philae made here the world's first successful soft landing on any asteroid or comet on 12th November 2014.

to resist its own weight beyond this scale of mass and size.[223] Mimas, a moon of Saturn, is currently the smallest known astronomical body, with a diameter of *c*. 400 km, which is "perfectly" round due to self-gravitation. The diameter of about 300 km appears to be the threshold limit between spherical and irregular shapes.[224] Asteroids, which are smaller than this, often look like "potatoes".[225] With the size of a few kilometers across, an asteroid or a nucleus of a comet can have a non-spherical and very irregular shape, see Fig. 4.11 above.

[223] David W. Hughes and George H. A. Cole, "The asteroidal sphericity Limit", Monthly Notices of the Royal Astronomical Society, Vol. 277, Issue 1 (1995), pp. 99-105, also available online at http://articles.adsabs.harvard.edu/full/1995MNRAS.277...99H see also http://www.spaceanswers.com/deep-space/what-is-the-minimum-size-a-celestial-body-canbecome-a-sphere/ (Both accessed 2016-02-23).

[224] Hughes and Cole (1995).

[225] Charles H. Lineweaver and Marc Norman, "The Potato Radius; a Lower Minimum Size for Dwarf Planets", *Proceedings of the 9th Australian Space Science Conference*, 2010, also available online at http://arxiv.org/abs/1004.1091 (accessed 2016-02-23).

Based on their mathematical and physical properties, there are good grounds to consider the circle or its three-dimensional counterpart the sphere as the most "natural" or "perfect" form of all. The Greek Neo-Platonist philosopher Proclus (AD 412–485) stated this very clearly: "The circle is the first, the most simple and the most perfect figure."[226] Proclus' opinion in this matter reflected those of previous philosophers such as Aristotle and Plato.[227] German naturalist Lorenz Oken held the opinion that most tissues in living matter consisted of spherical monads: "The sphere is, therefore, the most perfect form. […] The inorganic is angular, the organic spherical."[228] Astronomical and mythical models of the universe throughout history have used circles and spheres in describing the realms beyond our terrestrial world.[229] The sphere has seemed to be a natural shape belonging to the realm of the eternal and immutable heavens. Comets caused such a fear in ancient times because they seemed to break this apparent immutability of the heavens. As mentioned earlier, however, Kepler discovered that the real orbits of the planets were not perfect circles but almost circular ellipses instead. It is rather strange that to the end of his life, Galileo rejected the idea of a planet having an elliptical orbit.[230]

The Spiral

The spiral is a form so often seen in nature that even if I have not included it in the basic forms in this thesis, I have decided to write briefly about it for the sake of completeness. Some phenomena in non-living nature produce only temporary spirals like whirls in water or hurricanes and tornados in the atmosphere. Seashells, gastropods, ammonites or ram's horns have their spirals made from permanent hard materials, whereas some plants have soft spiral parts to attach themselves into other plants or structures. Some spiral forms in nature, such as the spiral galaxies, are vast, whereas some are extremely small, such as the spirochete bacterium, tobacco mosaic virus or DNA. Sometimes elegant spirals appear in surprising places, such as in beehives or crystals; see Fig. 4.12.

[226] Stefan Hildebrandt and Anthony Tromba, *Mathematics and Optimal Form*, 1985, p. 145.

[227] *Ibid*.

[228] Philip C. Ritterbush in G. S. Rousseau (ed.), *Organic Form: The Life of an Idea*, 1972, p. 43.

[229] Already in Sumerian culture, for example, there was a strong connotation of a circular universe. See Norriss S. Hetherington (ed.), *Cosmology: Historical, Literary, Philosophical, Religious, and Scientific Perspectives*, 1993, p. 46.

[230] Panofsky (1954), pp. 10-15.

Figure 4.12 Some bees build their hives in the form of a spiral; this example was built by Australian native stingless honeybee *Tetragonula carbonaria* (left).[231] A double interlaced triagonal spiral on a silicon carbon crystal, magnified x150 (right).

The Tree-like Form

In addition to the Euclidean geometric shapes just described, I want to treat also the tree-like form as an elementary or basic form in the context of this thesis. As the very name reveals, the tree-like form is easily perceived in visible and tangible nature familiar to us. The tree-like form is also called *dendritic* or *dendriform* after the Greek word δένδρον (*dendron*) "tree". A natural tree-like pattern (of non-organic origin) inside a rock called *dendrite* is seen in Fig. 4.1. This kind of branching shape is met in innumerable many living and non-living structures and phenomena of nature such as nerves, rivers or blood veins, lungs, or in strong electric discharge, minerals and crystals, to give just a few examples.[232]

There is also a large-scale (*c*. 2.4 x 1.6m) wall painting of a tree-like shape with a heavy trunk and a few branches in the cave of Lascaux, which, to my knowledge, is

[231] In colder climates, hexagonal cells are enclosed inside a round wall made of a grey paper-like mass. Such nests typically have a spherical form which minimizes the material needed for building the outer paper-like covering

[232] See, for example, Philip Ball, *Branches*, 2009. Unfortunately, I was unable to get full access to the book V. Fleury, J.-F. Goyet, and M. Léonetti (eds.), *Branching in Nature: Dynamics and Morphogenesis of Branching Structures*, 2001. Alchemists called the dendritic crystallizations of lead and silver in nitric acid "Arbor Saturni" and "Arbor Dianæ", respectively. Arbor is Latin for "tree".

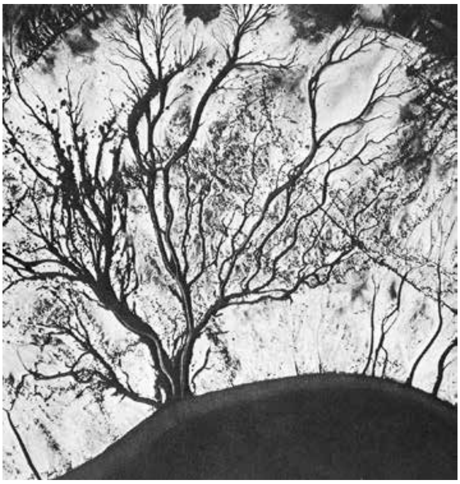


Figure 4.13. The estuary of the Colorado River, a high altitude aerial photograph from the 1950s.

the oldest known depiction of a tree-form.[233] This is also the only known depiction of such subject matter in all of Paleolithic parietal art.[234] This tree-like shape is located in a narrow place and is executed with a dark red colour onto a very uneven and curvy wall, making it difficult to photograph and reproduce in a proper manner. Hence, I have redrawn the shape with a brighter hue to better show the tree-like form along with the more prominent depiction of horses; see Fig. 4.14 (left).

The tree-like form is often used to illustrate otherwise non-visible relationships in, for example, biology. The *family tree* diagram can be used on very different scales from the lineage of single individuals within a family all the way up to presenting the complete genealogy of every known living or extinct species.[235] It is worth mentioning that the only image that Charles Darwin (1809–1882) included in his pre-eminent work *On the Origin of Species* (1859) was a schematic *tree of life*, representing the gradual descent and separation of the species from the older species.[236] Ernst Haeckel in

[233] Norbert Aujoulat, *Lascaux: Movement, Space, and Time*, 2005. Aujoulat (1946–2011) was very careful to not describe the shape *explicitly* as a tree; see, for example p. 130, Fig. 89: "a very large figure, the form of which is not unlike that of a tree […]" and p. 134: "[F]igure in this space is unique in Paleolithic parietal art. It takes the form of two large red branching lines, positioned vertically and facing each other […], little comment has been made on them." See also Mario Ruspoli, *Lascaux, the Final Photographs*, 1987, p. 118: "[B]ranching lines reminiscent of primitive vegetation, which André Leroi-Gourhan sees as symbolizing the branches of deer's antlers." In my opinion, the form looks very much like a stout tree and not at all like antlers. Leroi-Gourhan's antler interpretation is also rejected in Aujoulat (2005), p.134.

[234] Aujoulat (2005), p. 134.

[235] Such attempts have gained more momentum lately, after the developments in the databases of the cladistics and in the technical tools of interactive visualizations.

[236] Darwin presented the image of the branching structure as a foldout after p. 116 in his *On the Origin of Species*. In the beginning, there was possibly an alternative metaphor for the tree of life, namely "the coral of life", as was proposed by the German art historian Horst Bredekamp in his book *Darwins Korallen; Frühe Evolutionsmodelle und die Tradition der Naturgeschichte*, 2005.

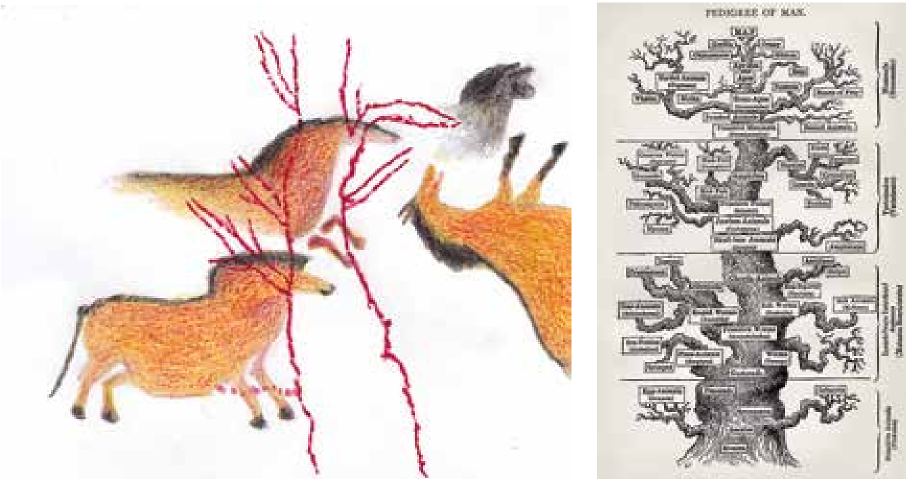


Figure 4.14 The earliest known depiction of a tree-like form is *c.* 17,000 years old from the cave of Lascaux, redrawn here by myself after Aujoulat 2005 (left), and the "Pedigree of Man" in the tree of life by Ernst Haeckel, 1879 (right).

particular, who was an energetic and influential proponent of Darwinian theory in Germany, sometimes used very naturalistic and almost tangible illustrations of the tree of life[237] in his books; see Fig. 4.14 (right). Of course, the idea of presenting relationships between things with a tree-like or other hierarchic structures was not completely novel in human thinking as such.[238] The family tree, for example, was first used to visually represent the family lineage of Jesus[239], then the family lineage of the sovereign, then the noble, the gentry, commoners, valuable stallions, and so forth, until Darwin revolutionized the whole scheme by replacing individuals with complete species.

An even older concept is the philosophical/religious idea of how all living and non-living things exist in the world, and how they are organized hierarchically in

[237] See, for example, Theodore W. Pietsch, *Trees of Life; A Visual History of Evolution*, 2012.

[238] Marion L. Kuntz and Paul G. Kuntz (eds.), *Jacob's Ladder and the Tree of Life; Concepts of Hierarchy and the Great Chain of Being*, 1988.

[239] Or, *The Tree of Jesse*, as it is called after Jesse, an ancestor of Jesus, mentioned in the Old Testament: "There shall come forth a rod out of the stem of Jesse, and a branch shall grow out of his roots and the Spirit of the Lord shall rest upon him." (Isaiah 11:1–2). See, for example, Roger Cook, *Tree of Life*, 1974, pp. 84–85.

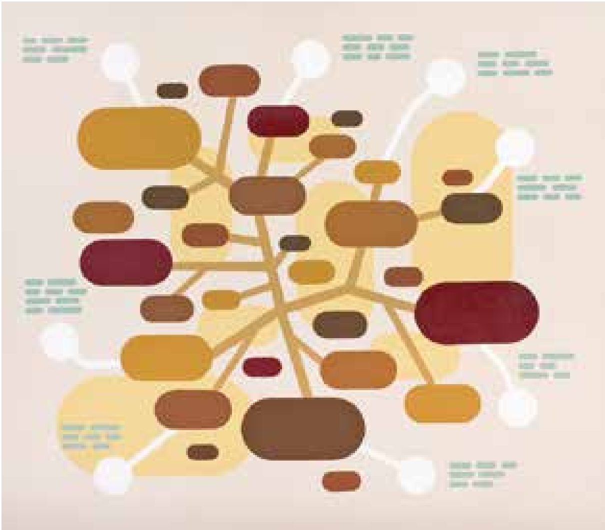


Figure 4.15 Markus Rissanen, [M75], Stemnotes, 70x80 cm, acrylic on canvas, 2004

a strictly linear way according to their "perfection".[240] This *scala naturae*, or the *great chain of being* begins from God and descends hierarchically to archangels, angels, humans, animals, plants, precious stones, metals, minerals and ends with substances such as gravel and sand.[241] Already Aristotle (384–322 BC) and later Carl von Linné (1707–1778) in his *Systema Naturæ* (1735) proposed similar types of linear "Ladder of Life" models for plants and animals, but their models were static and did not leave room for change of any type. Later the French naturalist Jean-Baptiste Lamarck (1744–1829) suggested that species could have evolved gradually over a long time, but his explanation of the inheritance of the acquired characteristics was false.[242]

The Greek philosopher Porphyry (AD 234–*c*. 305) developed Aristotelian linear hierarchy and transformed it into the form of a tree.[243] His model became popular during the Middle Ages along with the rise of scholasticism, and the schema bore

[240] Arthur O. Lovejoy, *The Great Chain of Being: A Study of the History of an Idea*, 1948 (1936). This seminal book established the discipline nowadays known simply as the history of ideas. In 1940 Lovejoy (1873–1962) founded the Journal of the *History of Ideas*.

[241] Kuntz & Kuntz (1988), and Lovejoy (1964), *passim*.

[242] See, for example, Alpheus Spring Packard, *Lamarck, the founder of Evolution*, 1901, and Peter J. Bowler, *Evolution: the History of an Idea*, 2003 (1984), pp. 82–89. The former book is also available online at https://archive.org/details/lamarckfoundere00packgoog (accessed 2016-06-22).

[243] See, for example, Manuel Lima, *The Book of Trees: Visualizing Branches of Knowledge*, 2014, Porphyrian Tree and Porphyry; p. 27–44.

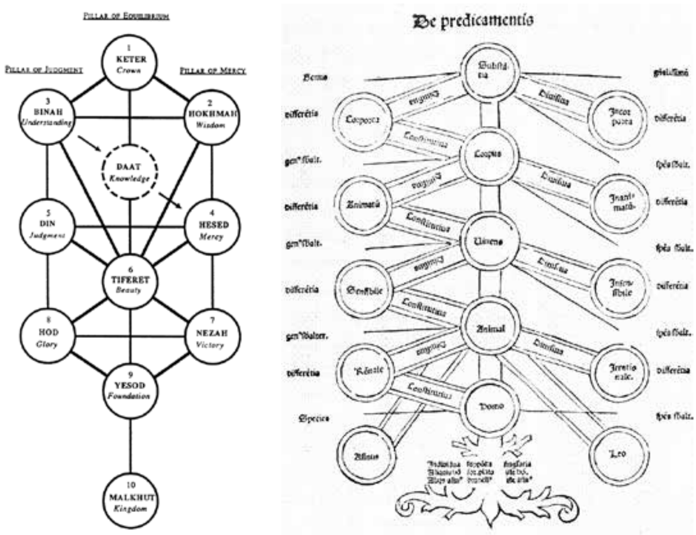


Figure 4.16 The Sefirotic Tree of Kabbalah, from Parpola 1993 (left), and the Porphyrian Tree 1498, from Bredekamp 2005 (right).

his name as the *Porphyrian Tree*; see Fig. 4.16 (right).[244] The genealogy of the metaphor of a tree in the history of science, religion and philosophy thus goes all the way – quite literally – into the *roots* of our western culture. One example of such a deeply-buried cultural loan is the religious Tree of Life theme inherited by Jewish and Christian religious mythologies from the older Assyrian culture; see Fig. 4.16 (left).[245] It can be argued that the Darwinian tree of life is also a relative to, and a descendant of, such philosophical, religious, and even esoteric tree-structures.

The branching structure resembling a tree or root system is often used as the visual scheme for many different things in all possible fields because it is an extremely versatile form.[246] The simple tree-model can be transformed further and widened into a network – or *rhizome* –, where the connections are more intertwined, clustered and complex than they are in the classical tree-model with only vertical connections.[247] Even the iconic idea of the Darwinian tree of life seems to have

[244] *Ibid.*, p. 27.

[245] Simo Parpola, “The Assyrian Tree of Life: Tracing the Origins of Jewish Monotheism and Greek Philosophy”, *Journal of Near Eastern Studies*, Vol. 52, No. 3 (July 1993), pp. 161-208.

[246] For example, Lima (2014) is an up-to-date survey of this wide subject.

[247] The Sefirotic Tree found in the Jewish Kabbalah is an example of a structure which resembles more a network than a tree. See Parpola (1993), pp. 169–189.

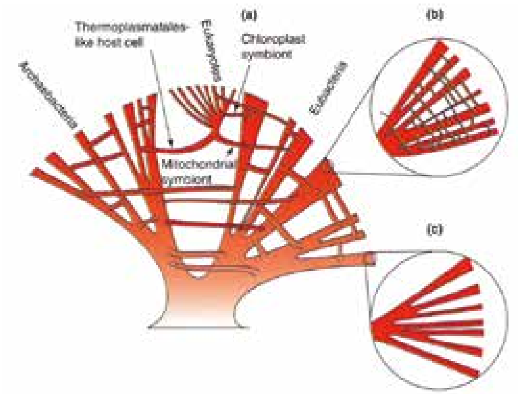


Figure 4.17 At the origin of life, the relations between species have turned out to resemble more a network of life than a tree of life. Image from McInerney, Cotton, and Pisani (2008).

soon turned into a "network of life"; see Fig. 4.17. For some decades, biologists have understood that there exists a phenomenon called *horizontal gene transfer* in the lower forms of life, in which the organisms not only pass genes "vertically" to their descendants but also exchange them "horizontally" with other species.[248] Observing such a phenomenon in the earliest forms of life is obviously a great challenge to taxonomical classification and even to the classic conception of species themselves. This tree-like structure is often characterized by a property that is often absent in the Euclidean shapes, namely *scale invariance*, which means that the structure of the object remains more or less the same independent of the scale on which we choose to study it. In other words, if we look more closely, the branches of a tree-like form will look more or less the same as the whole object: its details are miniature copies of the whole. Such a property is also called self-similarity and is an important property for a large class of geometric objects known as *fractals*.[249]

Fractals

In 1968 Aristid Lindenmayer (1925–1989), a Hungarian theoretical biologist living in the Netherlands, published a description of a system which is named after him, the *Lindenmayer-system*, or simply *L-system*.[250] This system describes

[248] James O. McInerney, James A. Cotton, and Davide Pisani, "The prokaryotic tree of life: past, present… and future?", *Trends in Ecology & Evolution*, Vol. 23, No. 5 (May 2008), pp. 276–281.

[249] Many objects of Euclidean geometry are also "self-similar". For example, no matter how much one zooms in on a straight line, its "parts" will always look the same as the "original" line. Also, a square can be divided into four smaller squares, for example, which can be divided into even smaller squares, etc. These squares can be considered as "parts" of the largest square and have the same shape as the original. This type of self-similarity is more trivial compared to the self-similarity in fractals.

[250] A. Lindenmayer, "Mathematical Models for Cellular Interaction in Development", *Journal of Theoretical Biology*, 18 (1968), pp. 280–315.

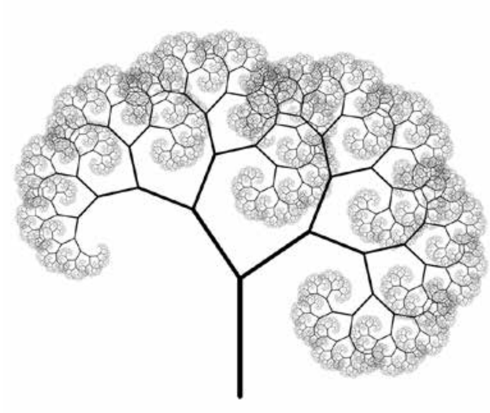


Figure 4.18 The “Pythagorean tree”, defined by a repeated bifurcation, is an example of a form produced by the L-system.

the way in which *recursion*, that is, the repeated use of a rule and the replacement of one element with a new (rule-produced) element, creates systematic forms with similarities to the forms of real plants. L-systems produce self-similar forms, and they can be used to create fractals, see Fig. 4.18.

As the term “fractal” suggests, a characteristic property of the class of such forms is that they are not “smooth” but typically rugged, wrinkled, or “fractured”. This property is connected to their dimension, which differs from that of the objects of Euclidean geometry. The circle, square, cube, or other examples of classical Euclidean geometry are zero-, one-, two- or three-dimensional objects. Properly defined, fractals, on the other hand, are objects with some characteristic part which has a *non-integer dimension*. Just as a smooth straight line occupies space only in one dimension, a wrinkled zigzag-type fractal occupies “more” space, that is, its dimension is more than one but less than two. According to mathematics, fractals have a certain, precisely-defined *Hausdorff–Besicovitch dimension*, which can have non-integer values such as 0.538 or 2.749 instead of 0, 1, 2, or 3, the only values objects of traditional geometry can have.[251] The perimeter curve, a round or straight line of the circle, square, or triangle, has the dimension 1 but the wrinkled perimeter of the Koch snowflake, for example, has a “fractured”, non-integer dimension of $\log 4 / \log 3 \approx 1.26186$; see Fig. 4.19. One technical definition of a fractal is “a set for which the Hausdorff–Besicovitch dimension strictly exceeds its topological dimension.”[252]

[251] The Hausdorff–Besicovitch dimension – or simply the Hausdorff measure – introduced in 1918 by Felix Hausdorff (1868–1942) and later improved by Abram S. Besicovitch (1891–1970), is used to define the dimension of the geometric object. Other measures for non-integer dimensions also exist, the older Minkowski–Bouligand (1904) measure and the more modern Assouad (1979) measure, for example.

[252] Benoit Mandelbrot, *The Fractal Geometry of Nature*, 1982; the definition of the term “fractal” is found on pp. 14–16.

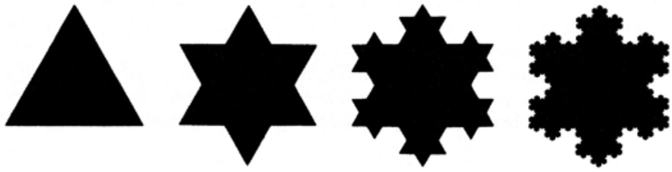


Figure 4.19 The first three iterations of one of the earliest exactly described fractal curves; the Koch snowflake published in 1904 by the Swedish mathematician Helge von Koch (1870–1924).

By the mid-18th century, mathematicians had already constructed these kinds geometric objects with strange properties, and for about the next hundred years such objects were considered as “pathological” or “monsters”.[253] One of the earliest examples of such precisely-defined “pathological curves” is the continuous but nowhere differentiable[254] Weierstrass function (1872)[255], which, as it happens, is also a self-similar fractal. See Fig. 4.20 below.

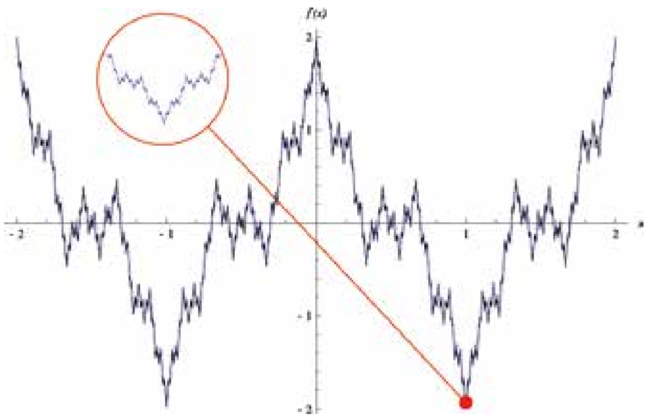


Figure 4.20.The Weierstrass function from 1872 is everywhere continuous but nowhere differentiable. Its self-similar structure is shown in the enlarged circle at left: the detail of the curve has the same characteristic shape as the overall form of the whole curve.

[253] See, for example, Freeman Dyson, “Characterizing Irregularity”, [a review of the book Mandelbrot 1977], *Science*, Vol. 200 (12 May 1978), pp. 677–678, or Edward Kasner and James R. Newman, *Mathematics and the Imagination*, 1968 (1940), with the appendix “Pathological Curves”, pp. 296—306, which makes no mention of he people who discovered and studied these “pathological” curves and after whom many of the curves are named.

[254] Mathematicians say that a curve is “differentiable” if to its every point a tangent can be unambiguously defined. *Differential calculus* is needed to conduct such an operation algebraically, hence the term “differentiable”. The Weierstrass function, on the other hand, is so “fractured” that it is impossible to define a tangent to a single point on it, hence “nowhere differentiable”.

[255] See, for example, Eric W. Weisstein “Weierstrass Function”, with a long list of references, at http://mathworld.wolfram.com/WeierstrassFunction.html (accessed 2016-07-07).

From the 1960s onwards Polish-born Franco-American mathematician Benoit Mandelbrot (1924–2010) developed the study and theory of "irregular" or "random" phenomena.[256] His 1967 paper "How Long Is the Coast of Britain?" was an important contribution to the study of "irregularity".[257] Mandelbrot proposed that the length of such a wrinkled curve could not be objectively defined, as it would depend on the precision of the measurement. If a ruler of 1000 km was used, the result would be a few times the measure at maximum, but if a ruler of 1m were used, then the length would become exhaustively long, and if such a curve as the coast of Britain were to be measured with the precision of 1mm, the length would appear practically infinite, as one would need to take into account not only the larger bays and capes as well as the boulders and stones, but even the tiniest pebbles and grains of sand.[258] Such a phenomenon is not observed for a smooth curve in the classic Euclidean manner.

Figure 4.21 A detail of a maple leaf with vivid autumn colours and a fractal-like structure; a photograph by the author.

Many fractal-like objects and phenomena have been found in nature in copious amounts since the 1960s. Nowadays fractal-like structures are recognized to exist in all kinds of objects and phenomena in nature. It seems nowadays almost self-evident that fractals form a more characteristic "language" of nature than the Euclidean forms do. Many fractal-like shapes and patterns are easily found in nature, even with the naked eye; see, for example, Fig. 4.21 above. Fractals were widely studied and quite popular in the 1980s and 1990s due to their strong visual nature and the development of computers, printers and displays at the time.

[256] One example of such a study is his essay "The Variation of Certain Speculative Prices" in the *Journal of Business* (Chicago) 36 (1963), pp. 394–419, reprinted in P. H. Cootner (ed.), *The Random Character of Stock market Prices*, 1963, pp. 297–337.

[257] "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension", *Science,* New Series, Vol. 156, No. 3775 (May 5, 1967), pp. 636–638.

[258] The English mathematician and meteorologist Lewis Fry Richardson (1881–1953) had earlier described this phenomenon, but his contribution remained mostly unnoticed. It seems that this idea by Richardson was published only posthumously in 1961. Mandelbrot, however, gave Richardson his due recognition in his 1967 paper and again in his book in 1982.

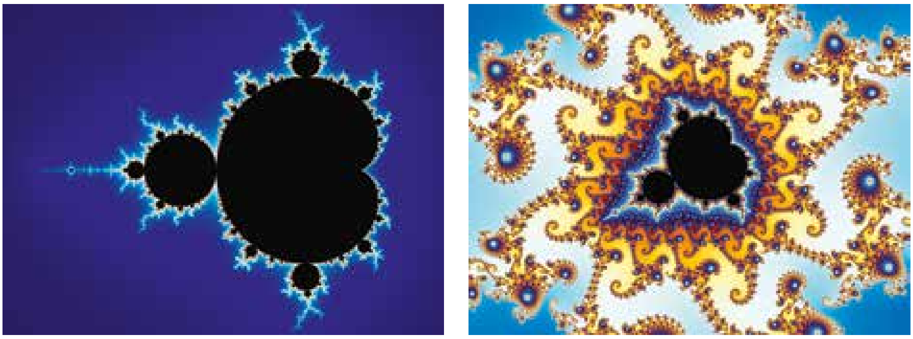


Figure 4.22 The Mandelbrot set is the area marked in black (left). A magnification of its boundary reveals an endless variation of forms; the shape of the original set in particular occurs again and again, on all levels of magnification (right).

The Mandelbrot Set

Mandelbrot coined the word "fractal" in 1975, and around 1978 the eponymous *Mandelbrot set* was discovered.[259] The Mandelbrot set is an object in the complex plane which has real and imaginary axes meeting perpendicularly in the centre.[260] One obtains from the initial value z_0 a series of complex number values z_0, z_1, z_2, z_3, z_4, z_5, … by the simple equation $z_0 \rightarrow z^2_0 + c = z_1$, which in turn gives the next value $z_1 \rightarrow z_1^2 + c = z_2$, and so forth. The parameter c is *constant*, meaning its value remains the same. The process always starts from the value $z_0 = 0$. What is

[259] The word "fractal" was introduced in 1975 in Mandelbrot's book *Les objects fractals: forme, hasard et dimension*, which was revised, expanded, and published in English in 1977 as *Fractals: Form, Chance and Dimension*, which in turn was further revised and expanded to *The Fractal Geometry of Nature* (1982). The events surrounding the discovery of the Mandelbrot set are not so straightforward, however. Prominent mathematicians Adrien Douady (1935–2006) and John H. Hubbard (b. 1945), who were the first to rigorously prove some of the most important mathematical properties of the set, gave it its current name in 1982. In 1978 Robert Brooks and Peter Matelski wrote a paper describing a similar set and included a crude image of its overall shape, but their paper was not circulated until the next year and was not published until 1981. It also seems that they did not realize the complexity or the importance of the set. On the other hand, Mandelbrot himself published his findings in December 26, 1980 in the *Annals of the New York Academy of Sciences*, and in a way his discovery can be seen as a justified culmination of his life-long research. See, for example, John Horgan, "Who Discovered the Mandelbrot Set?" in *Scientific American*, March 13, 2009, available online at http://www.scientificamerican.com/article/mandelbrot-set-1990-horgan/ (accessed 2016-06-27). Horgan's story appeared originally in the April 1990 issue of *Scientific American*.

[260] The word "complex" does not refer here to complexity in the sense of something being "complicated" but rather to complex numbers (a + b*i*), which are defined with two components: the real part a and the imaginary part b*i*, where *i* is the imaginary unit with the property $i^2 = -1$.

important in the process is the fact that values are defined in the two-dimensional complex plane and not in the one-dimensional line of real numbers. It is the parameter *c* which determines the end result of the process.

The Mandelbrot set, the black shape in Fig. 4.22 (left), is defined as the collection of all those values *c*, to which the value z remains bounded; that is, it does not grow beyond the finite limit in the repeated process $z \rightarrow z^2 + c$. If the initial value $z_0 = 0$ grows beyond all limits, or "escapes to infinity" in the process, then the value *c does not belong* to the Mandelbrot set.[261] This is where a computer steps in. A computer can calculate the values of as many points – or pixels – as wanted and draw the picture of the boundary of the set at any magnification, and all it takes is some programming, computing power, and time. One needs to test every pixel visible in the image to determine if it belongs to the set (pixel coloured black) or not (pixel having some other colour). By observing the growth speed outside the set, one can obtain additional information of the immediate neighbourhood of the set. The escape speed "to infinity" outside the set is often visualised with bright colours, as seen in Fig. 4.22 (right). [262]

The boundary of the Mandelbrot set is in some sense very, very, if not even "infinitely" complex. The principle of how the Mandelbrot set is defined is relatively simple – by mathematical standards – but the resulting object has been called "the most complicated object in mathematics".[263] The variety and complexity of forms within the boundary of the set never ends, no matter how much the boundary area is magnified. There are many types of spirals with different rotational symmetries:

[261] For example, with $c = -1$, the value z oscillates in a never-ending loop –1, 0, –1, 0, –1, 0 etc. With $c = 1 + i$, $z_0 = 0$, $z_1 = 1 + 3i$, $z_2 = -7 + 7i$, $z_3 = 1 - 97i$, $z_4 = -9407 - 193i$, etc. With $c = 2$, the initial value $z_0 = 0$ grows beyond all limits at an exponential speed: 0, 2, 6, 38, 1446, 2 090 918, 4 371 938 082 726, etc. The speed grows even greater as the value of the constant *c* moves away from zero. The size of a complex number (a + b*i*) is measured by its distance d from the centre (0,0). The distance d is the length of the hypotenuse of the right angled triangle (0,0), (a,0), (0,b), and the length is given by Pythagorean equation $d = \sqrt{|a|2 + |b|2}$. Even if the Mandelbrot set lies well within a circle of radius r = 2, this circle is not yet even a crude depiction of its shape; it just defines a "threshold value" beyond which the initial value $z_0 = 0$ grows with exponential speed.

[262] An alternative and more minimalistic black and white version can be obtained by dropping all the information about the "escape velocity" and leaving all such points simply white. See, for example, Heinz-Otto Peitgen and P. H. Richter, *The Beauty of Fractals*, 1986, or Heinz-Otto Peitgen and Dietmar Saupe (eds.), *The Science of Fractal Images*, 1988.

[263] John H. Hubbard as cited in A. K. Dewney, "Computer Recreations; a computer microscope zooms in for a look at the most complex object in mathematics", *Scientific American*, (August 1985), pp. 16–24, also available online, complete with its 1980s *zeitgeist* advertisements, at https://www.scientificamerican.com/article/mandelbrot-set/ (accessed 2016-07-01).

forms resembling seahorses, spider webs, thin filaments as in lightning, etc.[264] Such complexity must have suited Mandelbrot's personal taste well as he once stated in a rather terse fashion: "I hate minimal art."[265]

Organic Forms

It is fairly easy to describe a tree-shape or a fractal as an "organic form". Such forms seem to be "alive", unlike the circle, square, or triangle. Very likely, we intuitively feel that tree-shapes and self-similar fractals are more closely related to nature than the classic Euclidean shapes with their somewhat stagnated characters. I argue that it is because of this intuitive feeling that we are more or less amazed when first encountering such sharply defined forms of Euclidean geometry in nature. At least I personally had a strong and long-lasting feeling of amazement when seeing for the first time, for example, the objects depicted in Figs. 4.4–4.5, at how much they resemble the work of the human mind and hand. On the other hand, such examples show clearly that not all forms of nature are "organic forms".

That the concept of "organic form" itself is closely related to biology was no surprise to me, but the fact that the concept of organic form has – perhaps even most – of its "roots" in the field of *literary theory* was a surprise.[266] It was Aristotle who used the term "organic" first, but Plato himself had expressed a similar idea as he had Socrates state in *Phaedrus* that a fine discourse "should be organized, like a living being, with a body of its own, as it were, so as not to be headless or footless,

[264] Douady and Hubbard proved in 1982 that the Mandelbrot set is actually "simply connected", meaning it is one undivided area, such as a disk, in the sense that it has no separate islands or holes in its structure even if it *looks* exactly like the opposite. See Adrien Douady and John H. Hubbard, "Exploring the Mandelbrot set: The Orsay Notes", available online at http://www.math.cornell.edu/~hubbard/OrsayEnglish.pdf (accessed 2016-06-27). The Japanese mathematician Mitsuhiro Shishikura (b. 1960) proved in 1998 that the boundary of the Mandelbrot set has the Hausdorff–Besicovitch dimension of two, meaning that the line defining the boundary of the set is actually so wrinkled that it completely fills – a part of – the two-dimensional plane. See M. Shishikura, "The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets", *Annals of Mathematics* 147 (1998), pp. 225–267, a preprint of the paper is available at http://arxiv.org/pdf/math/9201282v1.pdf (accessed 2016-07-01). The Mandelbrot set is closely related to *Julia sets*, and *Fatou sets*, objects named after the French mathematicians Gaston Julia (1893–1978), and Pierre Fatou (1878–1929), who discovered and studied such sets in the beginning of the 20th century. As it happens, Mandelbrot was a student of Julia from 1945 to 1947.

[265] In Pfenninger and Shubik (eds.), *The Origins of Creativity*, 2001, p. 207. Mandelbrot continued with a sour description of a work from the 1960s by the American minimalist artist Carl André (b. 1935).

[266] G. S. Rousseau (ed.), *Organic Form: The Life of an Idea*, 1972.

Figure 4.23 Paul Klee, *Foliage*, Indian ink on paper, 24.2 x 30.7cm, 1931.

but to have a middle and members, composed in a fitting relation to each other and to the whole."[267] The modern formulation of the concept can be traced to German Romanticism, and to the theories of the English critic, poet, philosopher and literary theorist Samuel Taylor Coleridge (1772–1834).[268] Coleridge had not only written in his *Biographia Literaria* (1817) of ideas concerning organic form in the creative arts but explicitly of his own ideas about biology in his treatise [*Hints Towards the Formation of a More Comprehensive*] *Theory of Life* (1818) as well.[269]

Like Coleridge, the famous German poet Johann Wolfgang von Goethe (1749–1832) also published a treatise on biology, on the metamorphosis of plants, to be precise. I am in no position to estimate the extent of real influence these philosophical works of the era have had, for example, on further developments

267 G. N. Orsini, *The Ancient Roots of a Modern Idea*, in Rousseau (1972); the reference to Aristotle is on p. 10 and the reference to Plato's *Phaedrus* (246 C) the next page. G. S. Rousseau gives an illuminating example in his introduction (1972, p. 5) that the comparison should not be carried *too* far as we might end up asking, "what corresponds to the stomach in a tragedy or how do we find the liver of a comedy?" See also Charles I. Armstrong, *Romantic Organicism; From Idealist Origins to Ambivalent Afterlife*, 2003, p. 53.

268 See, for example, Philip C. Ritterbush in Rousseau (1972), p 40–53 or Armstrong (2003), Chapter 2: *Absolute Organicism in German Idealism: Kant, Fichte and Schelling* (pp. 13–29), and Chapter 4: *Organic Vagaries: Coleridge's Theoretical Work* (pp. 51–80).

269 Coleridge's *Biographia Literaria* and *Theory of Life* are available online, for example, at https://archive.org/details/biographialiter03unkngoog and at https://archive.org/details/hintstowardsform00colerich (both accessed 2016-07-29).

in biology. Goethe's theory of the metamorphosis of plants at least did not prove a success among professional biologists. The British plant morphologist, anatomist, historian and philosopher of biology, and the first woman botanist ever to become a Fellow of the Royal Society, Agnes Arber (1879–1960), describes, among other things, the reception of the theory in her 1946 introduction to Goethe's work and gives her opinion on how Goethe failed in a strict scientific sense.[270] Philip C. Ritterbush, a historian of biology, on the other hand, wrote without the tiniest speck of apparent irony that with his ideas concerning the organic form, Coleridge helped to "open a new chapter in the history of biology."[271]

I admit that there are simply too many roots, branches, and hidden rhizomes in the concept of the organic form to be followed more deeply in this thesis. Nonetheless, as a summary, it can be said that even if organic forms are not typically considered to have a visible position among Euclidean shapes, the branching tree-like form can, and I would argue, *should* be considered as a basic form due to the way it represents forms of nature not only in the perceptual but also the conceptual realm.

[270] "It was a defect of Goethe's amateur pursuit of science that he was too much attached to his personal notions and never attained the professional's hard-earned capacity for seeing his own work in due proportion in the general stream of thought. He himself defended thamateur standpoint, on the ground that the non-professional, being free from the obligation to strive after completeness of knowledge, is better able to reach a height from which he may gain a broad view. He failed, however, to realise that detailed knowledge, not limited to the worker's own special line, though it may seem of little value considered in itself, is yet essential as forming a framework of reference for general principles." Arber, p. 77, in her introduction to "Goethe's Botany: The Metamorphosis of Plants (1790) and Tobler's Ode to Nature (1782)", *Chronica Botanica*, Vol. 10, No. 2 (Summer 1946), pp. 63–126.

[271] Ritterbush in Rousseau (1972), p. 40.

5 | The Nature of Forms

In this chapter, I discuss how the study of the perception and evaluation of forms has been conducted. The usability of different forms and structures in depicting information is examined with regard to a variety of contexts. Here I also address the etymological and philosophical connection between *form* and *information* and touch upon the question of how the visualization of information might have originated in prehistory.

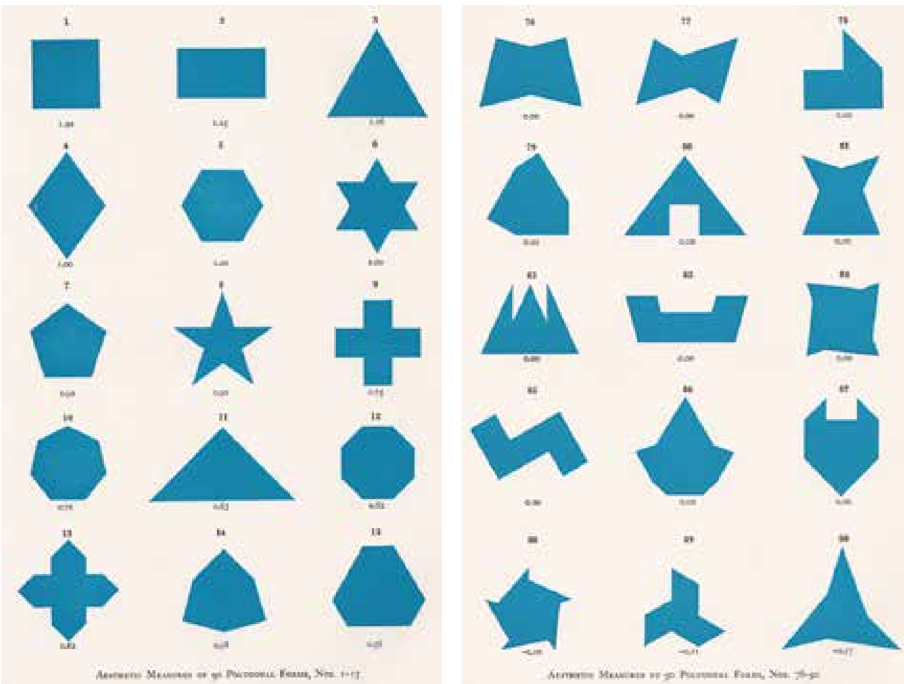


Figure 5.1 George D. Birkhoff, the aesthetic measures of 90 polygonal forms; Plates I and VII with 30 shapes from his 1933 book *Aesthetic Measure*.

Aesthetic Measures and Good Patterns

One of the leading American mathematicians of his time[272], George David Birkhoff (1884–1944), developed a mathematical theory for defining an "aesthetic measure" of an object in different fields, such as art, music, and poetry.[273] There are many difficulties in grasping the nature of forms in "scientific" or even in quantitative terms, and this is also the case with Birkhoff. From the perspective of a contemporary reader like myself, Birkhoff's equations seem to define more of a "perceptual" measure of shapes than the more subjective-sounding "aesthetic" measure.[274]

For evaluating polygons, Birkhoff had his scale descend from the maximum of 1.5 for the square to the minimum of –0.17 for the triangular shape, seen at the lower right corner in Fig. 5.1.[275] Birkhoff's shapes have been used almost as a kind of "standard set" of polygonal forms in many subsequent studies of aesthetic preferences, as they have a relatively long history with quantitative data, but for some reason, such shapes seem to not have been used in the study of forms if the topic is approached from the perspective of perception mechanism.[276] Birkhoff's examples included only polygonal forms and no round shapes, such as the circle. Nevertheless, the forms which scored highest in his scale are the square, the rectangle, and the (equilateral) triangle along with a rhombus, stars, and a cross with some regular polygons.

[272] Marston Morse, "George David Birkhoff and his Mathematical Work", *Bulletin of the American Mathematical Society*, Vol. 52, (1946), p. 389.

[273] George D. Birkhoff, *Aesthetic Measure*, 1933.

[274] Perceptual measure in the sense of how fast and correctly a shape is recognized.

[275] It can be argued that from the perspective of *arithmetic aesthetics*, it would have been easy to raise the whole scale by adding a constant of 0.5, thus making the values run from 2.0 down to 0.33. With this small operation, Birkhoff could have avoided the use of awkward-looking negative values for the aesthetic measure since that very negativity can give a shape an explicitly *ugly* connotation. This is not to say that his polygonal examples are beautiful *per se*, since in many cases they are quite the opposite, but the triangular shape, which reached the minimum –0.17 in his scale, is in my opinion not even the ugliest shape among his selected 90 polygons when compared to, for example, the triangular spiky "trident" shape in the middle row of the fourth column in Fig. 6.1, which reached only zero in Birkhoff's scale.

[276] See, for example, Frans Boselie and Emanuel Leeuwenberg, "Birkhoff revisited; Beauty as a Function of Effect and Means", *American Journal of Psychology*, Vol. 98, No. 1 (1985), pp. 1–39, H. Eysenck & M. Castle, "Training in Art as a Factor in the Determination of Preference Judgments for Polygons", *British Journal of Psychology*, Vol. 61 (1970), pp. 65–81, Christer Leijonhielm, *Colours, Forms, and Art; Studies in differential aesthetic psychology*, 1967, and Irvin L. Child, "Personal preferences as an expression of aesthetic sensitivity", *Journal of Personality*, Vol. 30, Issue 3, (Sept. 1962), pp. 496–512.

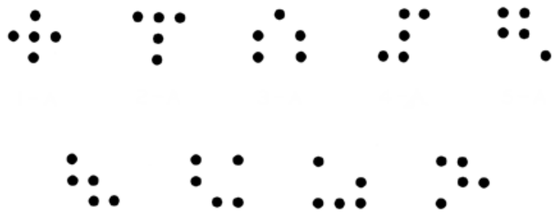


Figure 5.2 Patterns Wendell R. Garner used in his study of the "goodness" of patterns. In this image, the "goodness" decreases from left to right. The "cross" is considered the best, as it is the most symmetrical of all the patterns in this case.

Why is it that the same shapes always seem to pop up in different listings of "basic forms"? The American psychologist Wendell R. Garner (1921–2008) gave a simple answer in the title of his 1970 paper "Good Patterns Have Few Alternatives", even if he did not use the square, circle, or triangle in his research, but patterns made of five black dots placed in a 3x3 square matrix instead; see Fig. 5.2 above.[277]

Garner concisely sums up his main argument: "Poor patterns have many alternatives, good patterns have few alternatives, and the very best patterns are unique."[278] This simple truth implies that by randomly varying basic shapes, we easily end up with not-so-basic shapes or that by varying regular forms, we very likely end up with irregular forms. On the other hand, if we randomly vary forms which are already irregular, most probably we produce other irregular forms but no regular ones. In the words of Garner, we can agree that irregular shapes have many alternatives, good shapes have few alternatives, and the very best shapes are unique.

Birkhoff's theory is a good example of a Cartesian approach, where the problem or object under consideration is first broken into smaller parts which are separately solved one by one until the overall solution is then reached as the sum of these partial solutions. But there is also another way of approaching the study of reality, often named *holism*. In this approach, the parts are seen to exist in organic interconnection, "such that they cannot exist independently of the whole, or cannot be understood without reference to the whole, which is thus regarded as greater than the sum of its parts."[279]

[277] Wendell R. Garner, "Good Patterns Have Few Alternatives", *American Scientist*, Vol. 58 (1970), pp. 34–42.

[278] Garner (1970), p. 39.

[279] Part of the definition of the word "holism" in the *Oxford Dictionary of English*. The maxim "the whole is more than the sum of its parts" seems to have its origins in Plato: "the whole made up of the parts is a single form different from all the parts", (*Theatetus*, 204 A), as G. N. Orsini notes in G. S. Rousseau (1972), p. 20. Plato's formulation is much closer to the Gestalt psychologists' formulation than to the more famous but more lax maxim.

Gestalt Psychology

Especially in perception psychology, there has been an influential holistic movement known as *Gestalt psychology*, whose representatives, nevertheless, specifically stated that the whole is not "greater" than its parts, but rather fundamentally *different*.[280] The Austrian philosopher Christian von Ehrenfels (1859–1932) introduced the notion of "Gestalt" into psychology in 1890 in his essay Über Gestaltqualitäten (On Gestalt Qualities).[281] The most central representatives of the movement were Max Wertheimer (1880–1943), Kurt Koffka (1886–1941) and Wolfgang Köhler (1887–1967). They were all students of the German philosopher and psychologist Carl Stumpf (1848–1936), who in 1893 founded the Laboratory of Experimental Psychology in the Berlin University.[282]

Fig. 5.3 presents the Gestalt principles of how our mind groups objects according to proximity (a), similarity (b), good continuity (c), closure (d), common fate (e), symmetry (f), connection (g), and enclosure (h). Because of the proximity, or nearness, of the dots in (a), we more easily see three columns of dot-pairs

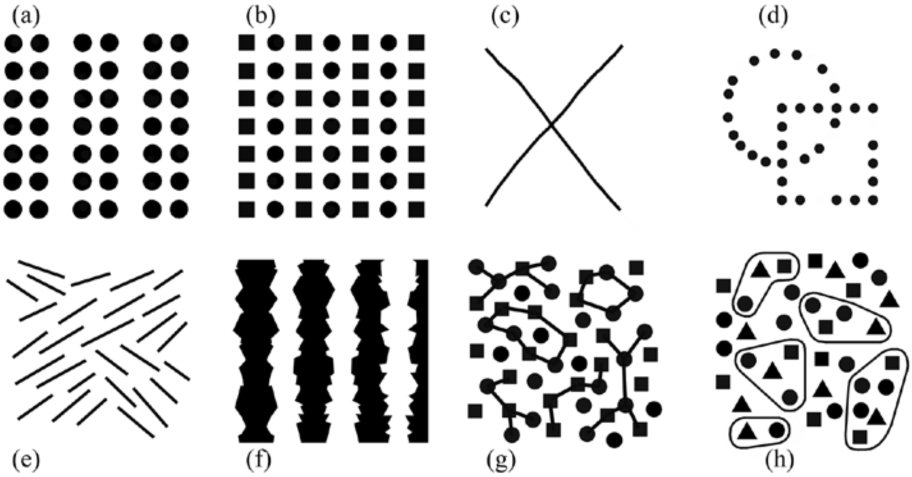


Figure 5.3 Gestalt laws of groupings.

[280] Mitchell G. Ash, *Gestalt Psychology in German Culture 1890–1967; Holism and the Quest for Objectivity*, 1995, p. 1.

[281] Ash (1995), p. 88.

[282] Johan Wagemans, "Historical and conceptual background: Gestalt theory", in J. Wagemans (ed.), *Oxford Handbook of Perceptual Organization*, 2015, pp. 4–5, also Ash (1998), *passim*.

than six columns of dots. In (b), we see the squares and the circles forming their own groups. In (c), we see two crossing lines with firmly continuing directions rather than two forms like > < with abruptly changing directions. In (d), we see what Gestalt psychologists called *good figures*: simple geometrical forms, such as the triangle, square, and circle, which cannot be reduced perceptually to any simpler, or "better" components.[283] The mind tends to repair and see such figures as complete and "closed", even when presented only in a very suggestive and incomplete manner.

In (e), we see elements grouped, as we perceive them to move in a common direction, that is, to have a "common fate" in the immediate future. We have a strong tendency to see simple images as separated in *figure* and in *ground*. [284] Sometimes there is ambiguity: what is figure and what is ground, and such ambiguity is present in (f), where the white and black areas seem to change their roles as ground and figure. In (f), we tend to see two black columns on white ground in the left side of the image and two white columns on black ground in the right side of the image because typically, figures (humans, animals, objects) possess a vertical mirror-symmetry but not ground. In (g), we see elements connected with lines as a group. In (h), we see enclosed elements as a group. Note how connecting (g) and enclosing (h) can override "weaker" principles of grouping, such as proximity and similarity.

The number and character of these laws varies slightly in different sources.[285] For example, cases (g) and (h) are not presented in historical context, and sometimes other laws, such as the "law of past experience" is included. Nevertheless, above these specific laws, there is a general principle of *Prägnanz*, German for "pithiness". In its general form, this principle states: "the perceptual field and objects in it take the most simple and impressive structure permitted by the given conditions."[286] In other words, we tend to organize our perceptions and experiences in as orderly, regular, symmetric and simple manner as possible. This principle can be observed also by comparing the images in Fig. 5.3; most probably it is the image (c) with

[283] Nicholas J. Wade and Michael T. Swanston, *Visual Perception: An Introduction*, 2001, pp. 78–79.

[284] One example with ambiguous figure-ground is the famous *Rubin vase illusion*, in which either a vase or two faces in profile is alternately seen. The Danish psychologist Edgar Rubin (1886–1951) published it in 1915. For the history of this illusion and Edgar Rubin, see, e.g. https://thepsychologist.bps.org.uk/volume-25/edition-1/looking-back-figure-and-ground-100 (accessed 2016-09-20), or see Ash (1995), pp.179–180.

[285] For example, five Gestalt grouping principles are mentioned in Wade & Swanston (2001), p. 78, four in Ian E. Gordon, *Theories of Visual Perception*, 2004, p. 17, and only three in Richard L. Gregory, *Eye and Brain; The Psychology of Seeing*, 1998, p. 4.

[286] Ash (1995), p. 224.

X that stands out from the rest as being the most terse, expressive and simplest among these eight images.[287]

The analysis of visual forms is not limited to already-existing shapes. There are many methods by which it is possible to construct well-determined forms starting from, for example, an arbitrary collection of separate points on a plane. One such method is to draw a circle with some fixed radius around every such point, and another method is to connect all the points with straight lines. Both of these methods certainly have their proper applications, but there are also other slightly more sophisticated methods. One such procedure is the *Voronoi tessellation*, also called the *Voronoi partition* or the *Voronoi diagram*.

Voronoi, Delaunay, and Harry Blum

The Voronoi diagram is a method which produces precise forms completely defined by any set of discrete points on a plane.[288] The process produces a kind of "spheres of influence" for these generating points, although the resulting forms are not round spheres but polygons, which are always convex. The Voronoi partition area (or cell) A_1 around a generating point P_1 is simply the collection of all the points of the plane, which are closer to P_1 than any other generating point P_x. In a normal Euclidean plane, one can define the border or "wall" between two neighbouring cells for generating points P_1 and P_2 by drawing a straight line connecting these points and drawing a second perpendicular line to the middle point of the first line. All the points of this perpendicular line are equidistant from P_1 and P_2. If the first lines connecting the generating points are depicted, one obtains the *Delaunay triangulation* (or -tessellation) for that particular set of points.[289] If the second lines, which are equidistant from two generating points, are depicted and connected, one obtains the Voronoi partition for that particular set of points; see Fig. 5.4. Although the construction of the Voronoi diagram may sound complicated to a layman, the resulting pattern is actually very intuitive for the mind and looks natural for the

[287] If there had been an image of a human face, an image of something repellent, for example, of a large hairy spider, something obscene, shocking, morally or politically questionable, such as a swastika, all such images would have most certainly gained even more attention than (c). Naturally, such examples do not belong under the question of how our visual perceptions are organized but what catches our attention because of some evolutionary or cultural reasons. Of course, the Gestalt laws of groupings have their origins in evolution, as they must have developed to help interpret the changing and chaotic visual environment in a reasonable way.

[288] Named after Russian-Ukrainian mathematician Georgy Voronoy (1868–1908). The tessellations are also known as *Dirichlet tessellations* after German mathematician Peter Gustav Lejeune Dirichlet (1805–1859). See, for example, Atsuyuki Okabe *et al.*, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 1992.

[289] *Ibid.*, p. 43 ff.

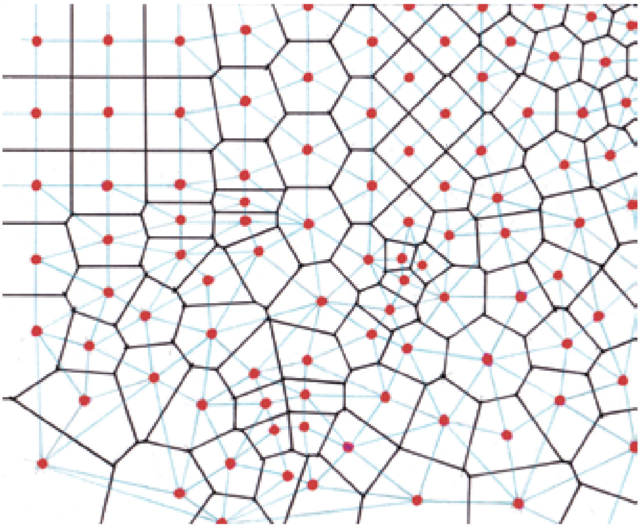


Figure 5.4 The Voronoi partition (black lines) for a set of generating points (red dots) on a plane. Tightly packed points have smaller cells around them, and more loosely packed points have larger cells around them. Blue lines depict the Delaunay triangulation.

eye. The resulting cells can be compared to areas governed by a city in the "middle" of each area, or to living cells with precisely one nucleus inside each cell. This partition is met encountered in a great variety of situations, from animal territories to all types of human activities.

A special type of Voronoi tessellation is the *Centroidal Voronoi Tessellation*, where each generating point is also the centre of mass of that particular cell.[290] Many patterns seen in nature are close to this particular type of Voronoi partition as its forms are typically an energetically close, often even exactly, optimal solution.[291] Voronoi tessellations and Delaunay triangulations are truly interdisciplinary concepts corresponding to an almost endless number of different patterns in living and non-living nature. Both concepts are used, for example, in anthropology, archaeology, astronomy, biology, cartography, chemistry, computational geometry, crystallography, ecology, forestry, geography, geology, linguistics, marketing, metallography, meteorology, physics, physiology, statistics, and in urban and regional planning.[292]

[290] See, for example, Qiang Du, Vance Faber, and Max Gunzburger, "Centroidal Voronoi Tessellations: Applications and Algorithms", *SIAM Review* [SIAM = Society for Industrial and Applied Mathematics], Vol. 41, No. 4 (1999), pp. 637–676, also available online at http://www.personal.psu.edu/staff/q/u/qud2/Res/Pre/dfg99sirv.pdf (accessed 2016-07-01).

[291] Qiang Du, Max Gunzburger, and Lili Ju, Advances in Studies and Applications of Centroidal Voronoi Tessellations, *Numerical Mathematics: Theory, Methods and Applications,* Vol. 3, No. 2 (May 2010), p. 121.

[292] Okabe *et al.*, 1992, p. 2.

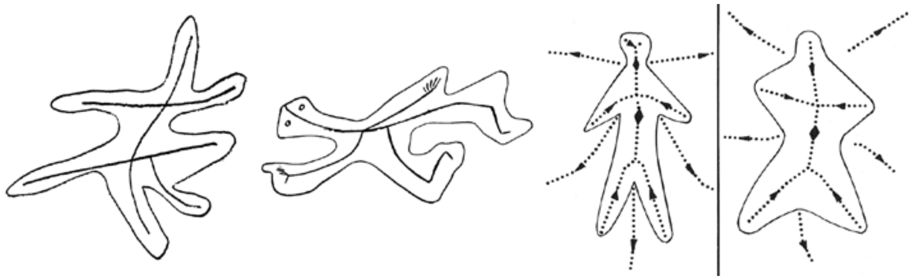


Figure 5.5 Paul Klee, an undated study (left) and a detail from *Spirits of the air,* pen-and-ink drawing (middle), both works from 1930, compared with two anthropomorphs and their "medial axis functions" from a theory of perception by biologist Harry Blum, 1964 (right).

In a sense, the Voronoi partition can also be applied in "the opposite direction". Just as the process in the Voronoi partition is defined by a set of fixed discrete points on a plane, a fixed shape on a plane also defines another complementary process. This process is known by different names, such as *topological skeleton*, *medial axis*, or *grassfire transformation*, and it was published in a seminal paper by mathematical biologist Harry Blum in 1967.[293] Blum worked his theory further, describing it again in a fairly long article (82 pages) in 1973.[294] The Voronoi process can be seen as a "shockwave" or "grassfire", expanding uniformly in all directions from separate points at constant speed. The resulting borders are always in the form of straight lines exactly where these "waves" meet and perhaps "phase out" or "cancel" each other. In Blum's model, the "shockwave" or "grassfire" propagates at constant speed *inwards* from every point of the outline of the shape. In a sense, both models describe the same situation; in the Voronoi partition, the perspective is "zoomed out", and the sources of the wave fronts lose their shape and appear point-like (see Fig. 5.4), whereas in the Blum model, the perspective is "zoomed in" to a single source of the wave front, which has some non-point-like shape, as seen, for example, in Fig. 5.5 (right).

[293] Harry Blum, "A Transformation for Extracting New Descriptors of Shape", in Weiant Wathen-Dunn (ed.), *Models for the Perception of Speech and Visual Form*, 1967, pp. 362–380. The publication was the proceedings of a symposium held in Boston in November 1964, sponsored by the Data Sciences Laboratory, Air Force Cambridge Research Laboratories, where Blum was working at the time.

[294] Harry Blum, "Biological Shape and Visual Science; Part I", *Journal of Theoretical Biology*, 38 (Feb. 1973), pp. 205–287. For some reason, Part II never appeared.

Just as the Voronoi partition is quite intuitive, ultimately, the same can be said of the Blum model. The resulting structure, i.e. the topological skeleton, or the medial axis of the form, reveals essential aspects of the shape. Not surprisingly, artists were using such skeleton schema before scientists.[295] One such revealing comparison can be made between drawings by Paul Klee and an illustration used by Blum; see Fig. 5.5. The skeleton model works well with concave forms such as animals or humans with protruding parts, but not so well with round convex shapes, which tend to "collapse" into a point in the process. Different types of skeleton models are applied nowadays, especially in fields related to computer vision, such as shape analysis, pattern recognition, fingerprint recognition, optical character recognition, machine learning, and so forth.[296]

The Formation of in-Formation

All visual information necessarily has a form. This fact is even included in the etymology of the word: *information* comes, via Old French, from the Latin word *forma*, meaning "mould or form".[297] In plastic, metallically electroplated CDs, this is true in a quite literal sense: the information is multiplied by moulding from one original "ideal" piece.[298]

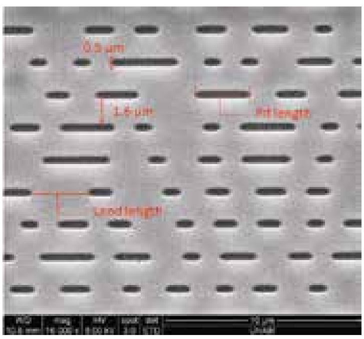


Figure 5.6 A scanning electron microscope (SEM) image of the surface of a CD. Distances between horizontal lines are 1.6 µm. Compare the scale of this artificial structure with that seen in the diatom in Fig. 4.5 (right).

295 As one might remark, even before artists, the skeleton schema was used by nature.

296 The field of computer vision and its use of skeleton models are simply too vast a subject to be discussed further in this thesis. For a general presentation, see, for example, https://en.wikipedia.org/wiki/Topological_skeleton or see Polina Golland and W. Eric. L. Grimson, "Fixed Topology Skeletons", in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, (CVPR'2000), which is also available online at http://people.csail.mit.edu/polina/papers/skeletons_cvpr00.pdf (both accessed 2016-07-09).

297 *The Concise Oxford Dictionary of Current English* (1995), pp. 531–532 and 697–698.

298 The manufacturing process is done using mouldings taken from one single master and not by burning every disc by laser, as I still believed until some months ago. See, for example, https://en.wikipedia.org/wiki/Compact_Disc_manufacturing (2016-08-01).

Figure 5.7 Aulis Blomstedt, the logo of the Museum of Finnish Architecture, designed in 1973 and used until the end of 2011 (left). An optical agent of information from the 2010s: a QR-code (right).

Such a simple fact as the divisibility of, for example, the number 60 can also be presented in visual form. The number 60 is divisible by 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30, which makes it a practical module in different fields of technical applications, such as architecture.[299] This divisibility was elegantly visualized in the logo of the Museum of Finnish Architecture, designed by Aulis Blomstedt in 1973, see Fig. 5.7 (left). Anachronistically, the logo's "binary" appearance more resembles some digital solutions of our time than the exaggerated round-edge "looks" of the 1970s.

But if we want to go back to the very beginning, where should we start looking for the origins of the visual recording of *information*? Naturally, all *figurative* images can be seen as visual recordings of information, but what if we search for recordings of information which is non-figurative, or "abstract" in its very character? All writing systems are also recordings of information, but some visual recordings, or "idea transmissions" go well beyond the invention of writing.[300]

The Bones of Information

At the border of modern-day Congo and Uganda, the Belgian geologist Jean de Heinzelin de Braucourt (1920–1998) found in 1950 a 10cm-long bone, which turned out to be a very old bone of a baboon.[301] The bone is marked with 167 (or 168) notches on its three sides. Even at the time, de Heinzelin de Braucourt

[299] The culture of using this number comes, of course, from the ancient Babylonians, from whom we have inherited the tradition of dividing one hour into 60 minutes and one minute into 60 seconds.

[300] Steven Roger Fisher, *A History of Writing*, 2001, pp. 11–27.

[301] Jean De Heinzelin de Braucourt, "Ishango", *Scientific American*, Vol. 206, Issue 6 (June 1962), pp. 105-116, Alexander Marshack, *The Roots of Civilization*, 1972, and Vladimir Pletser, "Does The Ishango Bone Indicate Knowledge of the Base 12?" The last one is a paper, so far only available online at http://arxiv.org/abs/1204.1019 (accessed 2016-08-22).

Figure 5.8 The Ishango bone from Congo, estimated age 20,000–25,000 years, depicted here from two sides. To better show the notches, I have digitally increased the brightness and contrast of this image.

suggested that the markings probably had something to do with mathematics. In 1964, American journalist and independent scholar Alexander Marshack (1918–2004) proposed the theory that some markings in Paleolithic objects could be lunar calendars.[302] Marshack developed his hypothesis further in his 1972 book *The Roots of Civilization*, which also included the Ishango bone.[303] I believe that there is much room left for arguments against most interpretations that Marshack presents of markings in Paleolithic objects. In addition, in his analysis, Marshack seems to find very few, if any, non-figuratively marked Paleolithic bones, or objects of other materials, *not* representing lunar calendars. In my opinion, this excessiveness throws a shadow of suspicion over Marshack's otherwise sound and original theory.[304] Nevertheless, ultimately, it actually doesn't matter for my argument whether such a lunar calendar hypothesis is correct in some particular case or not. In some cultures, such markings must have gained temporal continuity beyond single generations. It does not need to concern us here what kind of information was stored with such means. What is important to note is that such agents of information, either potential or actual, must have *existed* and must have been *available* for subsequent developments to take place.

For the purpose of my argument, it is enough to point out that at some point in prehistory, the habit of making systematic – and probably linear – markings *started* and gradually *evolved* into some kind of system of visual recordings or notations of some abstract phenomenon or other type of information. I argue that the

302 Alexander Marshack, "Lunar Notation on Upper Paleolithic Remains", *Science*, Vol. 146, Issue 3645 (Nov. 1964), pp. 743-745.

303 Alexander Marshack, *The Roots of Civilization, The Cognitive Beginnings of Man's First Art, Symbol and Notation*, New York: McGraw-Hill Book Company, 1972.

304 Scepticism towards such lunar calendar interpretations seems to be common among modern scholars in the field. See, for example, the article "Notation, Paleolithic", written by Steven Mithen in Brian M. Fagan (ed.), *The Oxford Companion to Archaeology*, 1996.

structures, which eventually developed into conceptual representations of nature and its phenomena, have their origins in this kind of notation or recording.[305] Here, I believe, we are dealing with the roots of such conceptual forms that represent nature. In them the human mind has found a way to depict the invisible, not in a symbolic but a quantitative manner. On many occasions, visualizations of information have utilized forms and structures that one may well describe as elementary. Perhaps the most effective and yet versatile tool for such work has been the orthogonal grid made of squares or of other types of rectangles.

From Lines to Grids, from Grids to Diagrams

Treatises on perspective were full of square grids after the theory of the mathematically correct linear perspective was invented in Renaissance Italy.[306] Nevertheless, the full potential of such a grid was revealed only after one direction of the grid marked some variable x (abscissa), and the other direction of the grid marked some variable y (ordinate), and these two variables were bound together by an algebraic equation. The result is a *graph*. René Descartes published such a system in 1637 in his book *La Géométrie* (as an appendix to his larger work *Discours de la méthode*), combining geometry with algebra by using co-ordinates.[307] This invention, nowadays known as *analytic geometry*, was ground-breaking in its time, as it combined numbers and equations with forms in an exact manner. It made solving many types of problems possible, at least in theory.

Descartes did not explicitly describe, or perhaps did not even understand, the best parts of his system, and it took some time before the practical aspects

[305] As opposed to mimetic representations of nature, both of which were introduced in the beginning of this thesis. A bone notched in this way might have formed an image of time that was developed – perhaps even accidentally – by a prehistoric mind, an image of such a natural phenomenon which could be observed and understood, but not depicted by imitation, because the very phenomenon, time, is *de facto* invisible and thus beyond all mimetic representation. Preconditions for such notations were naturally the development of counting and the making of readable marks in some durable material.

[306] This discovery is most often credited to the Florentine architect Filippo Brunelleschi (1377–1446), who is said to have demonstrated his discovery around 1425. See, for example, Hugh & Fleming (2010), and Samuel Y. Edgerton's, *The Heritage of Giotto's Geometry,* 1991, pp. 88–90 and *The Mirror, the Window, and the Telescope*, 2009, p. 39 ff. Edgerton proposed in *The Renaissance Rediscovery of Linear Perspective* (1975) that the ancient Romans had already correctly conceived the idea of linear perspective complete with vanishing point(s), which would thus only have been "rediscovered" later in Florence. This hypothesis is not universally accepted.

[307] Carl Boyer, *A History of Mathematics*, 1968, pp. 367–393, and Jakov Ljatker, *Descartes*, 1984, pp.179–231. Descartes did not, for example, depict right-angled co-ordinates.

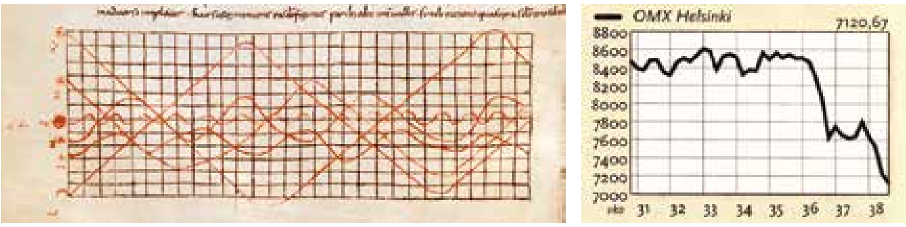


Figure 5.9 An illustration in the Latin manuscript Codex 347 from the Bern Burgerbibliothek, Switzerland (left). The graph in the parchment from the second half of the 9th century shows the apparent movements of the planets and accompanies a text by Pliny the Elder. A later 2008 example of another, less predictable "sub-prime" phenomenon is depicted with a similar conceptual form with time and one variable positioned in a right-angled co-ordinate system (right).

of analytic geometry became more obvious.[308] As D'Arcy Wentworth Thompson speculated: "I imagine that when Descartes conceived the method of coordinates, as a generalization from the proportional diagrams of the artist and the architect, and long before the immense possibilities of this analysis could be foreseen, he had in mind a very simple purpose: it was perhaps no more than to find a way of translating the form of a curve (as well as the position of a point) into numbers and into words."[309] This sounds much like what the maker of markings in the Ishango bone might have had in mind: to find a way to translate the things worth counting and recording them onto a physical token to be stored outside active human memory. Although the invention of the Cartesian coordinate system enabled the modern representation of functions, similar types of representations, especially of periodic spatio-temporal phenomena, had already been used (or at least experimented with), for example, in some medieval works; see Fig. 5.9 (left).[310]

Some Schoolmen in France and at the Merton Collage, Oxford, England, investigated the nature of such spatio-temporal phenomena and experimented with their graphic representations.[311] For example, Nicole Oresme (*c*. 1320–1382) in France produced such proto-co-ordinate images representing the idea of a *very*

[308] Boyer (1968), pp. 376–380. Boyer also discusses how Descartes did not create the analytic geometry all alone, but Pierre de Fermat (1601[07?]–1655) and François Viète (1540–1603) also played a big role.

[309] Thompson (1942), p. 1032.

[310] Other medieval manuscripts (MS) with similar graphs are mentioned, for example, in Harriet P. Lattin, "The Eleventh Century MS Munich 14436: Its Contribution to the History of Co-ordinates", *Isis*, Vol. 38, No. 3/4 (February 1948), pp. 205–225.

[311] Boyer, (1968), pp. 287–294, and Ljatker, (1984), pp. 190–196.

simple graphic representation of spatio-temporal phenomena.[312] The problem with medieval scholars was, of course, that to produce such graphic representations, exact measurements were needed, and as one modern author said of one medieval scholar: "Oxford's Richard Swineshead, for example, was ingenious in not dealing with exact measurements, but in avoiding the subject."[313]

As natural and as obvious as these graphs seem to us now, it took an amazingly long time for them to enter into wider use. Such graphic tools of information visualization remained rare until about the beginning of the 19th century.[314] Scientific journals began publishing graphs in the 1820s, and by the 1830s, graphs were regularly featured in scholarly journals.[315] One remarkable pioneer in the era was the Scottish engineer and political economist William Playfair (1759–1823), whose treatise *The Commercial and Political* Atlas (1786) is a milestone in the field of information visualization. Playfair has been credited, for example, for inventing the line-, bar-, and pie charts.[316]

From the Square to the Triangle and Beyond

Naturally, not all information representation or visualization is done with right-angled grids or co-ordinate systems. In Fig. 5.10 (left), for example, is a triangular nomogram graphically representing the melting temperatures of an alloy of three metals: lead (Pb), cadmium (Cd), and bismuth (Bi). Each point of the triangle corresponds to a unique proportion of a mixture of these three metals.

312 Howard Wainer, *Graphic Discovery*, 2005, p. 10–12, and Ljatker (1984), pp. 190–196.

313 Alfred W. Crosby, *The Measure of Reality; The Quantification and Western Society, 1250–1600*, (1997), p. 66.

314 Laura Tilling, *Early Experimental Graphs*, 1975, pp. 193–213.

315 Daniel R. Headrick, *When Information Came of Age*, 2000, p. 129.

316 Tilling (1975), p. 199–200, Edward Tufte, *The Visual Display of Quantitative Information*, 1993, pp. 32–34, 43–45, and Howard Wainer and Ian Spencer, *Graphic Discovery*, 2005, Chapters 2–4: 'Why Playfair', 'Who Was Playfair', 'William Playfair; A Daring Worthless Fellow', and 'Scaling the Heights (and Widths)', pp. 20–38. Even if it is sometimes stated that the famous Florence Nightingale (1820–1910) invented the pie chart, actually she did not *invent* it. Playfair had already published a pie chart in his *Statistical Breviary* in 1801, a work in which the fresh but tiny pie went largely unnoticed at the time. Nevertheless, Nightingale popularized the pie chart successfully in England in 1858, after the Crimean War (1853–1856). Her statistical observations of the causes of the casualties of the war caused political uneasiness: the majority of the casualties were not due to enemy fire, but to diseases and horrible hygienic conditions in the field hospitals. Another pioneer in information visualization, Charles-Joseph Minard (1781–1870), also used the circular pie chart in France exactly at the same time, in 1858.

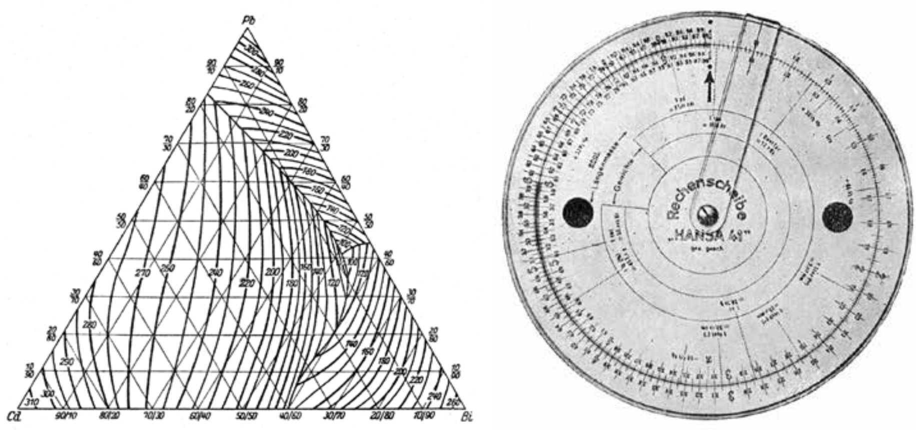


Figure 5.10 A triangular nomogram representing melting temperatures of an alloy of three metals (left), and a circular slide rule (right).

Each vertex marks a pure substance of only one metal, every edge corresponds to a mixture of only two metals, and the interior points correspond to an alloy of all three metals. The curvy contour lines correspond to the melting temperatures. To expand such a presentation to include a fourth substance would require a thee-dimensional tetrahedron with four vertices, and adding a fifth substance would require a four-dimensional tetrahedron with five vertices in a four-dimensional hyperspace. Hence, this model has its practical limitations, although its principle is beautiful in its logical simplicity.[317] Fig. 5.10 (right) shows a German circular slide rule *Hansa 41* – from *c.* 1941 – with two rotating logarithmic scales and a transparent cursor on top. Even the logarithmic scale doesn't have to be straight; it can be curled 360° to connect with itself and start the next round with tenfold magnified units, then next round with again tenfold magnified units, etc.

Periodic Tables, Snails, and Corkscrews

In many cases, the most used rectangular representation is not the only possible one. For several visualizations and graphic representations, there are other possible alternatives, each with its good and bad characteristics. Very seldom can a representation or visualization be said to be optimal in any absolute terms. One iconic conceptual model representing nature is the *periodic table*. It is often

[317] Of course, a whole book or a library with different combinations of several metals is feasible, or would have been feasible before the interactive digital era we are currently living in.

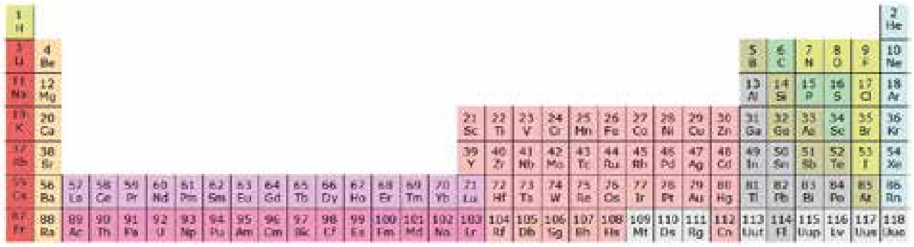


Figure 5.11 The traditional rectangular periodic table in its "long model", where lanthanides and actinides – marked here with pink shades – are incorporated in the main body.

depicted in the shape of a rectangular table consisting of small boxes representing the individual elements. Its usefulness comes from the fact that it is typically ordered in a way that elements with similar atomic structures – read: with similar chemical properties – are located in the same columns. Nevertheless, such an order in a two-dimensional rectangular matrix is not the only possible way to organize the elements.

The Latvian-American chemist Edward G. Mazurs (1894–1983) tried to record every serious two- or three-dimensional alternative model ever published of the periodic table. Mazurs presented these models in 1957 in a book which he self-published. In 1974 Alabama University Press published a second edition as *Graphic Representation of the Periodic System During One Hundred Years*. In this book, Mazurs divides the graphic representations of the periodic system into 146 divisions, classes, types, and subtypes, including altogether no less than *c*. 700 different models for organizing the elements.[318] Mazurs redrew all the images and "updated" old models to include new elements. His book is thus a collection of conceptual models, not a collection of historically accurate depictions. His work shows well how even such a simple and iconic conceptual structure like the periodic "table" is by no means chained to a rectangular format.

[318] Edward G. Mazurs collection with his original drawings, three-dimensional models, etc. are nowadays archived at the Chemical Heritage Foundation in Philadelphia. For some reason, all the links relating to the Mazurs collection seem to have disappeared when the foundation renewed their homepages July 2016. There are, however, some nice photographs of Mazurs' manuscripts taken by the chemist David Black available online at his chemistry-related blog at http://elementsunearthed.com/tag/edward-mazurs/ (accessed 2016-08-22).

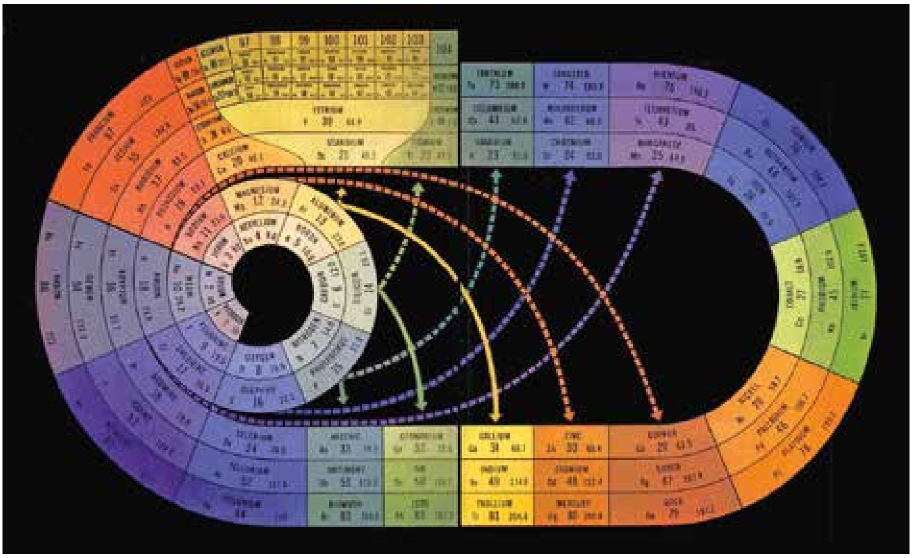


Figure 5.12 A periodic system from 1933 by John D. Clark. The coloured version shown here was a two-page centrefold in a feature article "The Atom", published in the LIFE Magazine May 16th, 1949.

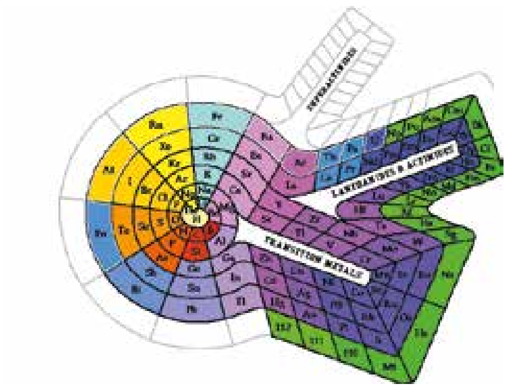


Figure 5.13 A modern coloured version of the "periodic snail" by chemist Theodor Benfey with finishing design touch by Joe Jacobs, the then-art-director of *Chemistry* magazine, where the model was first published in 1964.[319]

[319] The "snail", or "rabbit ears" model is presented also in Mazurs' book (p. 77), not in the 1964 model, as the book was originally published in 1957, but as an even earlier, 1947 model, which was erroneously credited in the book to Charles Janet and not to D. F. Stedman. Ironically, a 2010 photo by David Black of the Mazurs collection papers reveals that Mazurs had 'Stedman 1947' still correct in his manuscript; the error happened only later. Also, Benfey mentioned this error when he wrote of his "snail model" in 2009. Benfey's essay is available at www.scs.illinois.edu/~mainzv/HIST/bulletin_open_access/v34-2/v34-2%20p141-145.pdf (accessed 2016-08-28), originally published as Theodor Benfey, "The Biography of a Periodic Spiral", in the ***Bulletin of the History of Chemistry***, Vol. 34, No. 2 (2009), pp. 141–145.

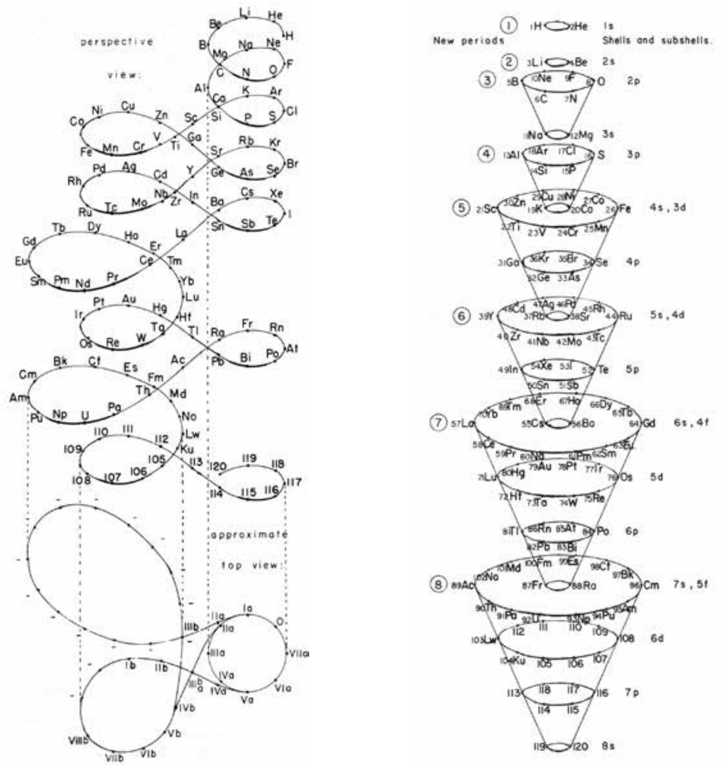


Figure 5.14 A three-dimensional model of the periodic system by Karl Schirmeisen from 1900, (left), and a three-dimensional model of con-centric circles proposed by Mazurs himself in 1967 (right), from Mazurs, 1974.

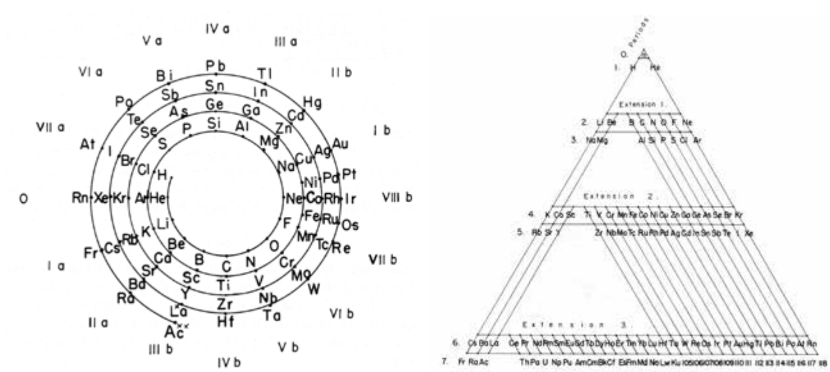


Figure 5.15 A circular spiral model of periodic system from 1887 by Flavitskii (left), and a triangular schema by Wagner and Booth from 1945 (right), from Mazurs, 1974.

6 | Forms Constructed – Structures Discovered

In this chapter, I analyse some tilings published by Dürer and Kepler. I also address the mathematical concepts of *periodicity*, *nonperiodicity*, *aperiodicity*, and *quasiperiodicity*. The principle of the *crystallographic restriction theorem* is also explained. Against this background, I give an account of the discoveries of the Penrose tilings and quasicrystals. Together these themes provide the necessary steps to introduce my own geometric discovery.

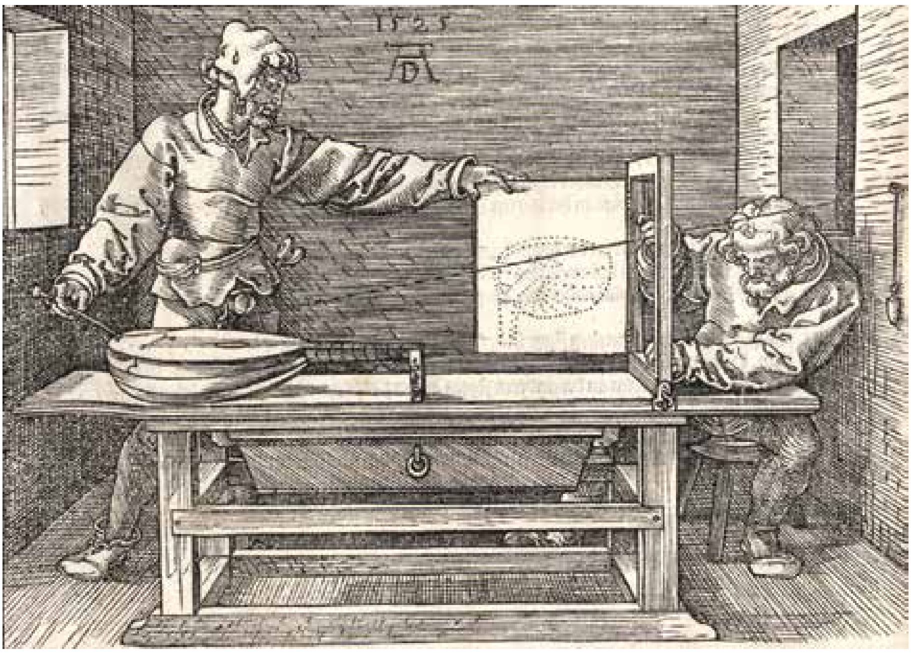


Figure 6.1 An image from Albrecht Dürer's book *Underweysung der Messung* (1525).

Straight Lineages

Albrecht Dürer published numerous depictions of geometric constructions and tilings in his *Underweysung der Messung* (1525), also known as *The Painter's Manual*.[320] One of his most famous images from this work depicts one particularly laborious way of constructing a perspective image, with the aid of a tight string, in this particular case, of a lute, see Fig. 6.1. About ninety years later Johannes Kepler actually repeated this experiment, albeit not with a round lute, but with a rectangular book.[321] Kepler's rationale for engaging in such laborious construction with a real string was to avoid "the arcane nature of light" from influencing the purely *geometric* aims of the experiment.[322]

Kepler was familiar with the geometric work of Dürer since he even mentions it in his *Astronomia Nova* (1609) and in his letters.[323] The Austrian art historian Otto Benesch (1896–1964) mentions that Dürer's *Underweysung der Messung* (1525) directly influenced not only Kepler's geometric constructions but also the language he used of the geometric objects in his *Messe Kunst Archimedis* (1615), which was a German translation of his *Nova Stereometria*, written in Latin one year earlier, in 1614.[324]

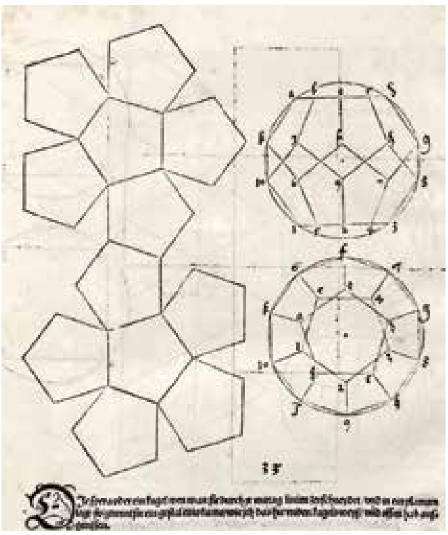


Figure 6.2 Albrecht Dürer, a regular dodecahedron folded open on a plane, *Underweysung der Messung* (1525).

320 Dürer, *The Painter's Manual*, 1977, pp. 157–169, or pp. 65–70 in the original 1525 edition at http://digital.slub-dresden.de/en/workview/dlf/17139/1/0/ (accessed 2015-09-18).

321 Stephen Straker, "Kepler, Tycho, and the 'Optical Part of Astronomy': the Genesis of Kepler's Theory of Pinhole Images", *Archive for History of Exact Sciences*, Vol. 24, Issue 4, (October 1981), pp. 286–287, footnote 37. Not only do Benesch and Straker but Panofsky also refers in his Galileo book (1954, p. 5, footnote 1) to the study by Leonard Olschki, *Geschichte der neusprachlichen wissenschaftlichen Literatur, Vol. 3,* 1927.

322 Straker (1981), p. 286.

323 Benesch (1965), p. 184, note 42. The same is mentioned also in Gyorgy Kepes, *The New Landscape in Art and Science*, 1956, p. 49.

324 Otto Benesch, *The Art of the Renaissance in Northern Europe*, 1965 (1945), p. 184, note 42. On the other hand, Benesch does not mention another, older German-language booklet,

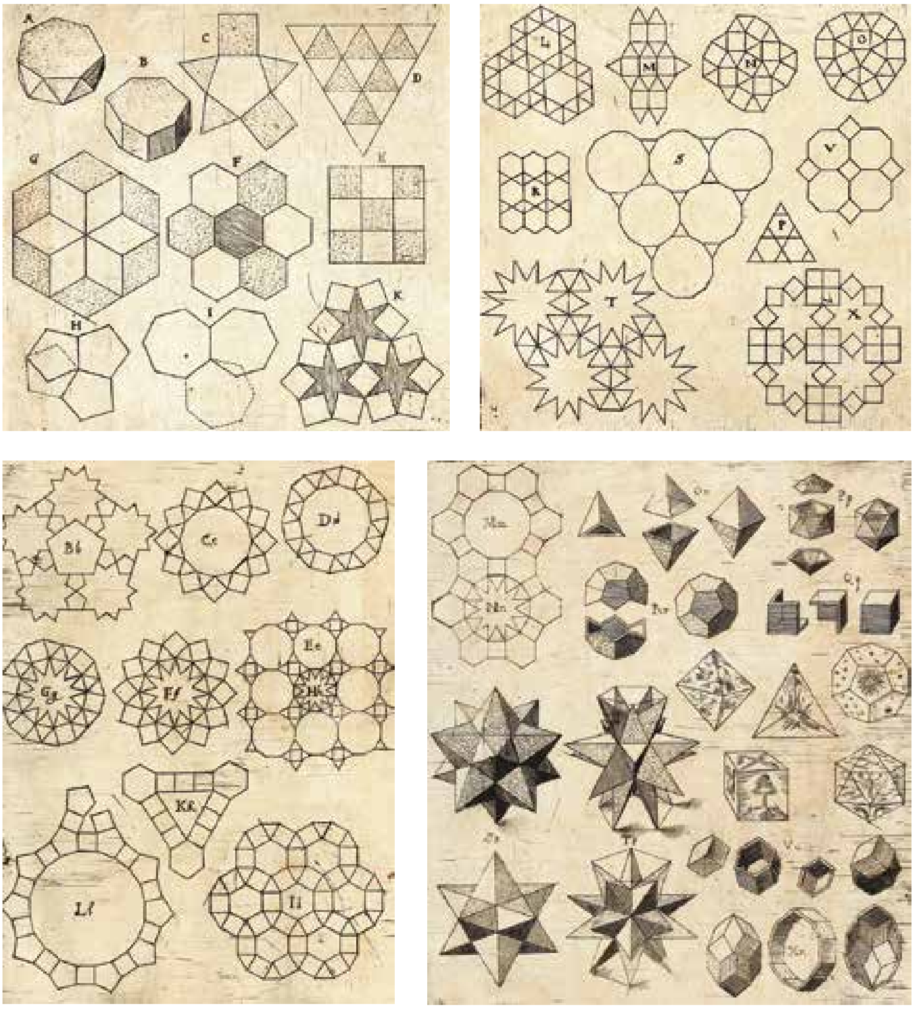


Figure 6.3 Johannes Kepler, tilings and polyhedrons from *Harmonices* Mundi (1619).

Dürer and Kepler had some similar interests in their geometric studies. In addition to three-dimensional constructions, they both studied two-dimensional shapes as well as methods connecting the two. Dürer's book, for example, included images of regular and semi-regular polyhedrons with their surfaces "left open" along their edges, folded flat on a page. One such image depicted the connected network of

Geometria Deutsch, *c*. 1484…1486, with no author, date, or printer given. This craftsman's manual for practical geometric work might have influenced some of Kepler's constructions just as it influenced Dürer, who apparently even took some of his constructions directly from it. See Walter L. Strauss' introduction in Dürer (1977), pp. 16–17. I am in no position to make further remarks about the development of geometric terms in the German language during the Renaissance. If possible, see, for example, Olschki (1919), mentioned a few footnotes ago.

twelve regular pentagons forming an unfolded dodecahedron; see Fig. 6.2.[325] Dürer appears to have been the first person to present a complete three-dimensional geometric object in such a flat-folded manner.[326]

The regular pentagon, or more specifically, the tilings in which it can be used, seems to have fascinated both Dürer and Kepler. From the perspective of tiling an infinite plane[327], there is indeed something fascinating in regular pentagons and in the fivefold rotational symmetry, which the regular pentagon inherently possess. Kepler was the first western writer to systematically study the problem of tilings formed by regular polygons.[328] Kepler published his study of tilings in *Harmonices Mundi* (1619).[329] The equilateral triangle, the square, and the regular hexagon can each tile the plane alone, as shown by Kepler in Fig. 6.3 (D, E, and F, respectively).

The regular pentagon, on the other hand, cannot tile the plane alone, the reason being that after connecting (edge-to-edge) three such pentagons, there is no room left for a fourth pentagon to share a common vertex without overlapping; see Fig. 6.3 (H). The same holds for any regular polygon with seven or more sides: after connecting two such polygons (edge-to-edge), there is no room left for a third polygon to share a common vertex without overlapping; see Fig. 6.3 (I) for heptagons. Only three regular polygons, the triangle, the square, and the hexagon,

325 Dürer (1525), pp. 142–157, Fig. 33.

326 Donald W. Crowe, "Albrecht Dürer and the Regular Pentagon", in István Hargittai (ed.), *Fivefold Symmetry*, 1992, pp. 465–488. Crowe mentions Max Steck's *Dürers Gestaltlehre der Mathematik und der bildenden Künste*, Halle (Saale): Max Niemeyer Verlag, 1948, as a comprehensive study of Dürer's mathematical work complete with a 448-item bibliography of commentaries up to 1948. See also Staigmüller (1891).

327 In mathematics, strictly speaking, a tiling refers to the complete covering of an infinite plane with no gaps or overlapping allowed. A finite covered area is called a patch.

328 Field (1988), p. 105. In the Islamic world, geometric tilings have been extremely popular for centuries, but in addition to the *Topkapı Scroll* in the Topkapı Palace, Istanbul, and "MS Persian 196" in the Bibliothèque nationale, Paris, very few ancient Islamic literary sources of the theory of tilings have survived. See Gülru Necipoğlu, *The Topkapı Scroll,* 1995, and *The Arts of Ornamental Geometry: A Persian Compendium* […], 2016. Two scholars in the field, Carol Bier and Jean-Marc Castera, confirmed this rarity to me also in private communications: in an email from 2016-08-25 (C.B.) and in a discussion in Paris from 2014-03-28 (J-M.C.).

329 Many images of his tilings are provided as unnumbered sheets in the second book (*Liber II*) of his *Harmonices Mundi*. There seem to have been serious problems with these separate image sheets in the printer's house. Apparently, there were plans for only two separate sheets, but in the Smithsonian Library copy, also available online, for example, at https://archive.org/details/ioanniskepplerih00kepl (accessed 2015-10-10), there are four separate one-side printed sheets for these images. In the Smithsonian copy, the separate sheets are placed after p. 58 (*Liber II*), when in fact they should have been placed after p. 52, for after this page the pagination jumps to 55. In the lower middle corners of pp. 52 // 55, there is even a small inscription – *Hic inserantur figurae ex // aere foliis 53–54* – explicitly stating that the intention was to have two copper-plate prints attached there as pages 53–54.

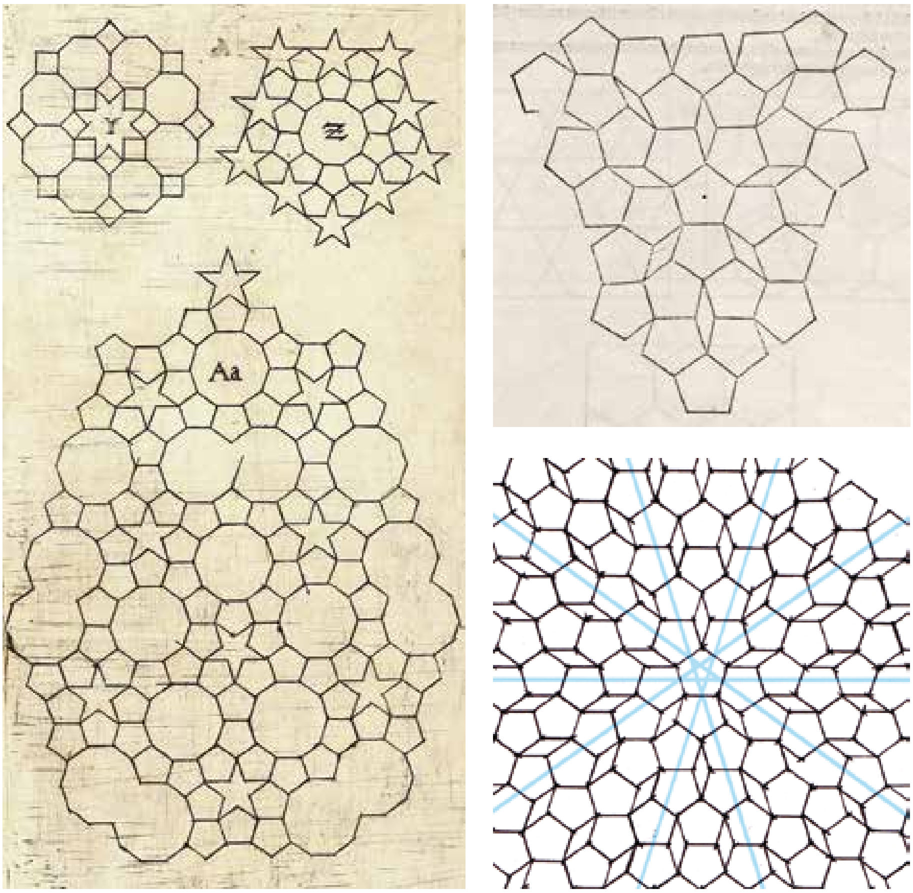


Figure 6.4 Johannes Kepler, patterns “Y”, “Z”, and “Aa” from *Harmonices Mundi*, 1619 (left), Albrecht Dürer, a pattern made of pentagons and rhombuses, 1525 (top right), and a solution suggested by the author to continue Dürer’s patch into a complete tiling (bottom right).

are able to cover a flat surface by themselves.[330] If regular polygons are mixed, more possibilities arise; see Fig. 6.3 (L, M, N, O, P, R, S, V, Ee, Ii, and Mm). If regular but non-convex “star” polygons are allowed, even more possibilities arise; see Fig. 6.3 (T, X, Hh, and Nn). On one occasion, Kepler chose to use even more complicated polygons in his tilings; see “Aa” in Fig. 6.4 (above left).

For some of Kepler’s images, it is not easy to say how the pattern is supposed to be extended further. In fact, some of his patterns are *impossible* to extend to cover the whole plane using only tiles depicted in the specific patch. For example, in

[330] Branko Grünbaum and G. C. Shephard: *Tilings and Patterns*, 1987, pp. 58–64.

the patch (Z) seen in Fig. 6.4, a deadlock arises after 5-sided stars.[331] To continue tiling, additional shapes are needed, such as triangles or rhombuses. Dürer (1525) combined thin rhombuses with pentagons to attain a system capable of tiling the whole plane; see Fig. 6.4 (top right).

It is fairly easy to see how Dürer's pattern may be continued to cover the whole plane. I present one solution in Fig. 6.4 (bottom right): ten V-shaped "wedges" (pale blue lines) meet at the centre. Five of them overlap at their tips, forming a pentagram inside the pentagon that is located in the centre. This overlapping is allowed, as these V-shaped "wedges" are only provisional markings and not part of the actual tiling, which constitutes of pentagons and rhombuses only. I also utilized a similar solution by using "wedges" in my first tiling system, which is presented shortly in Chapter 7 (see Fig. 7.3) and in a more detailed manner in Appendix A. The pattern inside these V-shapes, however, is periodic, and it repeats monotonously away from the centre, *ad nauseam*.

Both tilings in Fig. 6.4 by Dürer and Kepler seem to suggest a tiling with one point providing a centre for *globally* 5-fold rotational symmetry. Naturally, every regular pentagon constitutes a *locally* 5-fold rotationally symmetric pattern, albeit a small one. Note that the central point in Kepler's pattern is *not* in the dodecagon with letters "Aa" but inside the 5-sided star surrounded by five dodecagons. This central star is symmetrically surrounded by five dodecagons and five shapes, which are a fusion of two regular dodecagons, resembling an unopened peanut. This "peanut-shape" of the fourth tile apparently amused Kepler, as he explicitly called it a "monster".[332] Kepler didn't tell how he thought the "Aa" pattern should be continued. Nevertheless, using only the four types of tiles shown in Fig. 6.4 (bottom left), it is possible to continue the pattern "Aa to cover the whole plane.[333]

[331] The 5-sided star seen in the image is also called pentagram or pentacle if drawn with five crossing lines in a continuous way. Ancient Pythagoreans used the pentagram as their secret symbol, and for them it symbolized "health"; See Heath (1921), p. 161. This symbol and its meaning to the Pythagoreans was mentioned, for example, in *The Clouds*, a comedy by Aristophanes (*c*. 446–*c*. 386 BC), which premiered in 423 BC. The reference to the pentagram is in *The Clouds*, verse 609; see Heath (1921), p. 161. The Pythagoreans were not the only ones to study or use the pentagram in antiquity or since then. Pentagram has been used extensively as a magic symbol, sometimes standing "upwards" on two legs, sometimes pointing "downwards". In this latter position, especially when drawn inside a circle, the pentagram is a symbol often associated with occultism, black magic, and Satanism.

[332] Kepler (1619), p. 52.

[333] Grünbaum and Shephard (1987) thoroughly describe one solution for extending the pattern "Aa" to cover the whole plane and analyse how it can be done (p. 57 and pp. 88–90). They also discuss some other possible solutions. To go further with these solutions here would take considerable space and does not serve the purpose of this thesis.

There is a *relevant* difference between these two tilings, or patterns, using pentagons by Dürer and Kepler. The simplest continuation suggested by Dürer's pattern has its pentagons as its only finite areas, where any locally fivefold rotationally symmetric pattern exists. Kepler's pattern, on the other hand, suggests in addition to mere pentagons at least two other kinds of slightly larger areas that also have locally fivefold rotationally symmetric structures. Surrounding every dodecagon with ten pentagons makes one such patch, and surrounding every five-sided star with five-plus-five pentagons makes another such patch. In addition, one such patch with a five-sided star is clearly destined to become the centre of a tiling possessing a globally fivefold rotational symmetric structure.

Four Small Leaps for Mankind

Rotationally symmetric tilings are beautiful and fascinating as such, but are they of any interest to a modern person in general, or to this research in particular? The answer is affirmative: one main result of my study is a certain geometric discovery with a lineage leading up to it from the tilings by Dürer and Kepler, presented in Fig. 6.4. There are four steps connecting the tilings by Dürer and Kepler to my discovery: 1) the crystallographic restriction theorem and crystals, 2) the Penrose tilings, 3) Dan Shechtman and the quasicrystals, and 4) my first tiling system, *Hex Rosa*, which later developed into the even more sophisticated tiling system, *Sub Rosa*, and it is to this discovery that I am referring. The Sub Rosa tiling system emerged and evolved during and as a part of my doctorate studies between 2011 and 2015.

I will next go through these four steps. The Hex Rosa system is described in Appendix A.[334] A chronological, and somewhat more personal description of the development of the Sub Rosa system is provided in Chapter 7, written in as simple and non-technical language as possible. A more technical description of the Sub Rosa system and mathematically rigorous proof of its main properties are provided in Appendix B.[335] In order to give a decent description of these steps, I must introduce some concepts which are necessary to understand the nature of the outcome. I ask patience from my readers who are not trained in mathematics. On the other hand, I ask patience from those who are trained in the field, as I know well my own limitations in presenting clearly, and, I sincerely hope, correctly some necessary mathematical concepts.

[334] Appendix A is a reproduction of an article originally published in the peer-reviewed *Bridges Finland Conference Proceedings*, 2016, pp. 209–216.

[335] Appendix B is a reproduction of an article originally published in the peer-reviewed *Discrete & Computational Geometry*, Vol. 55, Issue 4, June 2016, pp. 972–996.

Crystals and the Crystallographic Restriction Theorem

Crystals have been appreciated since time immemorial for their beauty and natural or assumed supernatural properties. In addition to aesthetic pleasure, crystals have provided a fruitful platform for the study of symmetric structures. Like plants and animals, crystals also appear in great variety, but due to their simple atomic structure, they articulate certain geometric rules and regularities are exceptionally well. Unlike plants and animals that we easily see in our everyday life, well-shaped crystals are rare. The main reason for this inconspicuousness is their relatively small size.[336] When one sees a well-developed crystal, it may look amazingly "unnatural" and "artificial". It may appear like Mother Nature had been playing with these geometric building blocks in some distant past, but then changed her mind and decided to put most of her creative energy into producing plants and animals instead. But rules and regularities in animals and plants seem to come from other rationales than in crystals. The microscopic structure of a crystal is made of periodically repeating atoms.

In mathematics, a pattern is periodic if some translation maps it to itself with absolutely no rotation allowed. In plain English: if a pattern is copied on a transparent sheet and this sheet can be moved, without rotating it, in such a way that the copied pattern and the original pattern still correspond perfectly

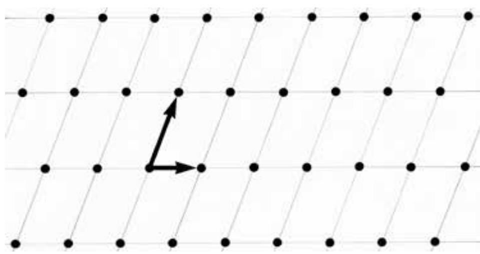


Figure 6.5 A two-dimensional lattice with its repeating parallelograms. The actual lattice consists of points only, not of lines or parallelograms. The basic vectors are marked here with arrows. Integer multiplications and combinations of these vectors give all possible locations for the points of the lattice. The angle between and the lengths of the basic vectors can be arbitrary.

[336] Exceptionally large crystals do also exist: the world's largest known crystals are 10–12 meters long and weigh about 50 tons. These giant selenite crystals were found in 2000 in *Cueva de los Cristales*, "Cave of the Crystals", located in Naica, Mexico. See, for example, http://ngm.nationalgeographic.com/2008/11/crystal-giants/shea-text/1 http://news.bbc.co.uk/2/hi/science/nature/8466493.stm or http://www.naica.com.mx/english/index.htm. (all three accessed 2015-09-22)

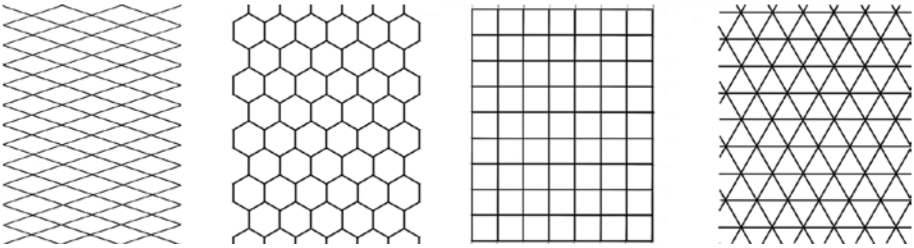


Figure. 6.6 Periodically repeating patterns made of rhombuses, regular hexagons, squares, and equilateral triangles having twofold, threefold, fourfold, and sixfold rotational symmetries, respectively. Note again that here it is also the shared vertex and *not* the centre of the shape that is assumed to be the point around which the rotation is assumed.

one-to-one, just as they did before the moving operation, the pattern can be termed periodic. The smallest distance in which this is possible is the period of the pattern in that particular direction; see Fig. 6.5. In a one-dimensional line, one direction is enough to define all such movements; in a two-dimensional plane, two independent directions are enough to define all such movements, and in three-dimensional space, three independent directions are enough to define all such movements. The simplest such patterns consist of points only. Such a geometric point structure is called a *lattice*, and its repeating "fundamental blocks" in one dimension are lines, in two dimensions *parallelograms*, and in three dimensions *parallelepipeds*.[337] One basic vector forms one edge of such a block, and its length is the period in that particular direction; see Fig. 6.5.

For all such periodically repeating patterns, there exists an important fact of nature, formulated in the so-called *crystallographic restriction theorem*, which says that the only possible rotational symmetries in such periodically repeating point systems, including physical crystals, are either "onefold", twofold, threefold, fourfold, or sixfold; no other values are possible.[338]

[337] The six faces of a parallelepiped are parallelograms. The word parallelepiped is composed of the Greek words παράλληλος + επίπεδος (a body with) "parallel planes".

[338] See, for example, Martin J. Buerger, *Introduction to Crystal Geometry*, 1971, pp. 20–22; 'Restrictions on the rotation angle'. H. S. M. Coxeter (1907–2003) presented a simple geometric proof in his *Introduction to Geometry*, 1962, pp. 60–61, which he credited to the British geologist and crystallographer William Barlow (1845–1934). Strangely enough, in Martin Gardner's *Penrose Tiles to Trapdoor Ciphers*, 1989, p. 27, the proof is credited, via the mathematician John Conway (b. 1937), not to the crystallographer William Barlow, but to the mathematician Peter Barlow (1776–1862). In geometry, a pattern is twofold, threefold, fourfold, …or *n*-fold rotationally symmetric if the pattern can be turned 180°, 120°, 90°, …or 360/*n* degrees, respectively, and the result looks the same, that is, with the same orientation, as it had before the rotation. In the rest of this book, I will omit the "onefold" case as it corresponds with one rotation of 360° or 0°, equivalent to no rotation at all.

These four cases correspond to a pattern having the rhombus, the regular hexagon, the square, or the equilateral triangle, respectively, as its “fundamental block” of translational repetition; see Fig. 6.6. In these cases, the pattern has not just one “centre” of global rotational symmetry but an infinite number of identical “centres”. Every vertex of any such a pattern can be assumed to be the centre around which the whole plane can be rotated and the pattern still looks the same. There is also a direct connection between the shape of such a fundamental block and the fact that these shapes, that is, the rhombus, equilateral triangle, square and regular hexagon, are the only regular polygons that can tile the plane by themselves.

As the name “crystallographic restriction theorem” indicates, its history is connected to the study of crystals. In crystals, this restriction is a direct result of the fact that all crystals have a periodically repeating internal structure made of molecules or single atoms. John G. Burke (1917–1989), a physical metallurgist and a historian of science, wrote in his *Origins of the Science of Crystals* (1966): “Crystals cannot have axes of five-fold symmetry, as do some flowers, nor axes of seven-fold symmetry, because the units that compose them must fill space. Five-sided blocks, for example, whose ends are regular pentagons, when fitted together would leave voids between them. Such units do not meet the requirement that space must be filled.”[339] The same was stated clearly also in a 1962 publication by the International Union of Crystallography: “Fivefold axes, common in botany (flower petals) and zoology (starfish) do not occur in crystals; nor are there axes of 7th or higher order.”[340] As all crystals are inherently periodic, they cannot have such “forbidden”, or “non-crystallographic” global rotational symmetry, that is, a global rotational symmetry for any other value than $n = 2, 3, 4$, or 6. The theorem can be used also in another direction: if some pattern *has* a centre of n-fold global rotational symmetry for any other value than $n = 2, 3, 4$, or 6, the pattern cannot be periodic and the centre is the only of its kind, or, in other words, the centre is unique in some “absolute” sense.

Against All Intuitions: Aperiodic Tilings

The idea of periodic structures seems intuitively clear and simple for the human mind. But as natural, simple, and common periodically repeating structures may be in our artificial environments and even in invisible microscopic structures, there are also other types of structures with some amount of order and repetition.

Every tiling in mathematics belongs to one of the two main categories: *periodic* or *nonperiodic*. These categories are mutually exclusive: a tiling is either periodic or

[339] Burke (1966), p. 3.

[340] Ewald (1962), p. 21.

nonperiodic. These categories also include all possible tilings; that is, there exist no tilings outside of these two categories. Nevertheless, these two categories are not the only categories used. The different types of tiles from which a tiling is constructed are called *prototiles*. Unlike in real life, in mathematics after we have fixed our set of prototiles, there is an infinite stockpile of them available to do the actual tiling. Some sets of prototiles enable both periodic and nonperiodic tilings. The regular hexagon, for example, enables only one tiling: the iconic honeycomb-pattern that is obviously periodic; see Fig. 6.6. The square, on the other hand, enables many different tilings, periodic and nonperiodic. The regular square-grid is clearly a periodic tiling, but its rows can be smoothly slid any arbitrary amount horizontally, and if we slide the rows in fractional or irrational amounts, differing in each row according to some nonperiodic rule, the resulting tiling is a nonperiodic tiling by squares. Thus, the most typical square-grid pattern is not a logical necessity; it is only an attractive option in its simplicity.

In 1961, research on abstract logic led the Chinese-American mathematician, philosopher and logician Hao Wang (1921–1995) to study some very general rules of tiling theory. Wang formulated the *Tiling Problem*, which asks: given a set of tiles, does there exist an algorithm or standard procedure for deciding, in a finite number of steps, whether this set tiles the plane? [341] In his research, Wang introduced square tiles with very simple rules defining the way that these squares can connect edge-to-edge with the other squares. Like two dominoes, which must have identical markings on their edges in order to connect, so also these square *Wang tiles*, or *dominoes*, have their edges marked with, for example, colours or numbers. These tiles can be moved by translation only – no rotations or mirror-reflections allowed – and they must connect edge-to-edge to four other squares with similarly coloured, or numbered edges always meeting. Wang proved that if any set of tiles, not just Wang tiles, can tile periodically in one direction, then the set can also tile periodically in the other direction, thus making the whole tiling periodic.[342] Based on this conclusion, Wang made the reasonable conjecture that if a set of tiles can tile the plane, it can tile the plane in a periodic manner. Wang was not able to prove this conjecture, which is equivalent to saying that there does not exist any set of tiles which can tile the plane *only* in a non-periodic manner. This conjecture seemed reasonable, as nobody had ever constructed a set of tiles with such a property.

[341] Hao Wang, “Proving Theorems by Pattern Recognition. Part II”, *Bell System Technical Journal* 40 (1961), pp. 1–41, or see Grünbaum & Shephard (1987), chapter 11, pp. 583–608; ‘Wang tiles’.

[342] Wang (1961), or see Bruno Durand, “Tilings and Quasiperiodicity”, p. 67 in *Automata, Languages and Programming*, (1997), pp. 65–75.

The situation changed in 1966 when Wang's student Robert Berger (b. 1938) proved that there does in fact exist a set of 20426 different Wang tiles which admits only a non-periodic tiling of the plane.[343] Berger's discovery was remarkable for it was not only the very first *aperiodic*[344] set ever found, but it also provided a definitive negative answer to the Tiling Problem, meaning that given a set of tiles, there is no algorithm or standard procedure for deciding, in a finite number of steps, whether this set tiles the plane.[345] Berger assumed that it was possible to find a smaller aperiodic set than 20426 prototiles to tile the plane, and he managed to reduce the number of required Wang tiles in that same year to 104.[346] Hans Läuchli reduced the number further to 40 (althoughbeit he did not publish his result until 1975), and Raphael M. Robinson (1911–1995) dropped the number first to 35 in 1971, and then to 24 in 1977.[347] Robert Ammann (1946–1994), a programmer and reclusive amateur mathematician who had already made other important contributions to the theory of tilings reduced this number further to 16 in 1980s.[348] The Finnish mathematician Jarkko Kari discovered a new technique to tackle Wang tiles, and in 1996 he published a set of 14 Wang tiles.[349] Using Kari's technique, Karel Culik managed to reduce the number further to 13 in the same year.[350] In June 2015 Emmanuel Jeandel and Michael Rao published a new aperiodic set of only 11 Wang tiles.[351] If their result is correct, the smallest possible set of Wang tiles has been found.

343 Robert Berger, "The Undecidability of the Domino Problem", *Memoirs of the American Mathematical Society*, No. 66 (1966), or see Grünbaum & Shephard (1987), pp. 584.

344 A set of prototiles, which enables only nonperiodic tiling is *aperiodic*. The resulting tiling, strictly speaking is not an "aperiodic tiling"; it is a nonperiodic tiling. Aperiodicity is not a property of a tiling (pattern), but a property of the set of prototiles that allows only nonperiodic tiling(s). See Grünbaum & Shephard (1987), pp. 520, 582 and 602–604.

345 Berger (1966).

346 Grünbaum & Shephard (1987), p. 584.

347 *Ibid*. pp. 589–593.

348 The rather sad story of Robert Ammann's life is told by Marjorie Senechal, "The Mysterious Mr. Ammann", in *The Mathematical Intelligencer*, Vol. 26, No. 4 (2004), pp. 10–21. His most important discovery in the theory of tilings is called *Ammann bars*. See, for example, Grünbaum & Shephard (1987), pp. 547–582.

349 Jarkko Kari, "A small aperiodic set of Wang tiles", *Discrete Mathematics* 160, (1996), pp. 259–264.

350 Karel Culik II, "An aperiodic set of 13 Wang tiles", *Discrete Mathematics* 160, (1996), pp. 245–251.

351 Emmanuel Jeandel and Michael Rao, "An aperiodic set of 11 Wang tiles", available so far only online at https://arxiv.org/abs/1506.06492 (accessed 2015-06-06). Jeandel and Rao announced that as their set contains 11 Wang tiles with 4 colours, it is the smallest possible set as no Wang set with fewer than 11 tiles is aperiodic, and no set with fewer than 4 colours is aperiodic.

From the Squares to the Rhombuses

The study of aperiodic tiles grew further in the 1970s and was no longer restricted to Wang tiles. Robinson discovered an aperiodic set of six tiles in 1971 and Ammann another aperiodic set of six tiles in 1977.[352] Both of these were based on modified squares but with a different type of connection rules than those for Wang tiles.[353] In 1977 Ammann developed no less than five new aperiodic sets of tiles, three of which were sets of only two tiles, but these discoveries were only published ten years later in 1987 by Grünbaum and Shephard.[354] All these aperiodic sets were based either on squares or on other modified rectangular shapes with their edges in two perpendicular directions, the only exception to this perpendicularity being Ammann's fifth aperiodic set, which used the square (90°, 90°) and a rhombus (45°, 135°) with specific connection restrictions applied to all edges.[355] After a further contribution in 1982 by F. P. M Beenker[356], this particular square-rhombus tiling is nowadays known as the Ammann-Beenker tiling.

An important discovery in the theory of tilings took place in 1974 when the British mathematical physicist Sir Roger Penrose developed his first aperiodic set of six tiles.[357] Penrose's system was especially interesting because it was not based on modified squares or perpendicular rectangles, but on a fivefold rotational symmetric structure that greatly resembled Kepler's pentagonal pattern presented in the 1619 *Harmonices Mundi*.

[352] Raphael M. Robinson, "Undecidability and nonperiodicity for tilings of the plane", *Inventiones Mathematicae* Vol. 12, Issue 3 (1971), pp. 177–209, and Grünbaum & Shephard (1987), pp. 525–530.

[353] *Ibid*. pp. 525–530.

[354] Grünbaum & Shephard (1987), pp. 525–530 and 550–558.

[355] *Ibid*., pp. 557–558.

[356] F. P. M. Beenker, Algebraic Theory of Non-periodic Tilings of the Plane by Two Simple Building Blocks: a Square and a Rhombus, 1982.

[357] See Roger Penrose, "The Role of Aesthetics in Pure and Applied Mathematical Research", *Bulletin of the Institute of Mathematics and Its Applications*, 10, No. 7/8 (1974), pp. 266–271, reproduced in P. J. Steinhardt and S. Ostlund (eds.), *The Physics of Quasicrystals* (1987), or see Martin Gardner, "Extraordinary non-periodic tiling that enriches the theory of tiles", in *Scientific American* 236 (January 1977), pp. 110–121, or see Roger Penrose, "Pentaplexity", in *The Mathematical Intelligencer*, Vol. 2, No. 1, (1979), pp 32-37, published originally in Cambridge University's Archimedeans fraternity magazine *Eureka*, No. 39, (1978), pp. 16–22, or see once again; Grünbaum & Shephard (1987), pp. 531–548.

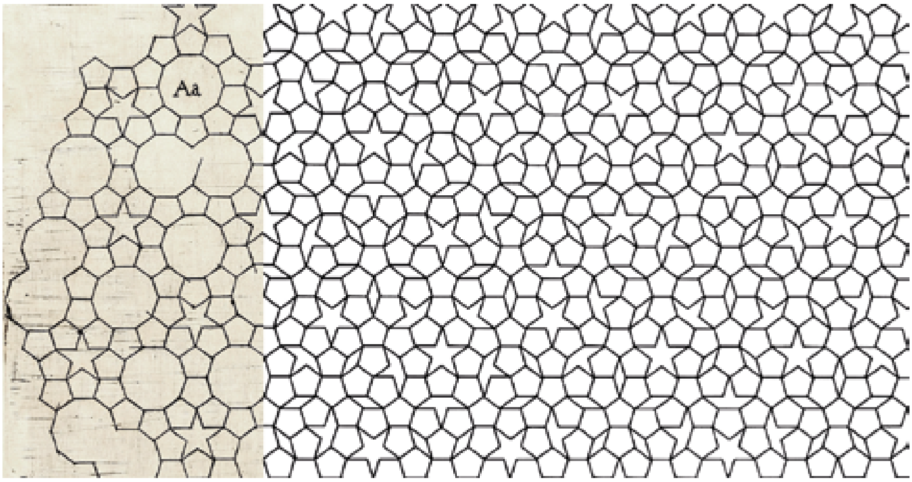


Figure 6.7 A juxtaposition of Kepler's pattern "Aa" from his *Harmonices Mundi*, 1619 (left), and Penrose's nonperiodic "stellar" tiling from 1974 (right). I have scaled and positioned these two images to make the seamless continuation between them more apparent.

As can be seen in Fig. 6.7, there is a close similarity between Kepler's and Penrose's patterns. Penrose wrote of this similarity in 2013:

> *Of more direct (although largely unconscious) influence was my much earlier acquaintance with Johannes Kepler's very remarkable 1619 picture exhibiting various non-crystallographic tiling patterns. I did not have these in mind when I first found my own aperiodic sets early in 1974, but I had seen Kepler's designs many years before, and I believe that they strongly influenced my attitude to the fruitfulness of pentagonal tilings. It was only some years after 1974 that I realized the extraordinarily close relationship between Kepler's configuration 'Aa' and my own pentagonal tiling, where the configuration of line segments constituting Kepler's entire configuration, without any exceptions, can be found within those of my own pentagonal tiling. I have always been intrigued as to how Kepler intended his pattern 'Aa' to be continued, and I believe it is quite within plausible possibility that he had in mind some sort of hierarchical continuation similar to that of my own pattern.*[358]

[358] Roger Penrose's foreword in Michael Baake and Uwe Grimm (eds.), *Aperiodic Order, Volume 1: A Mathematical Invitation*, Cambridge University Press, 2013, p. xi.

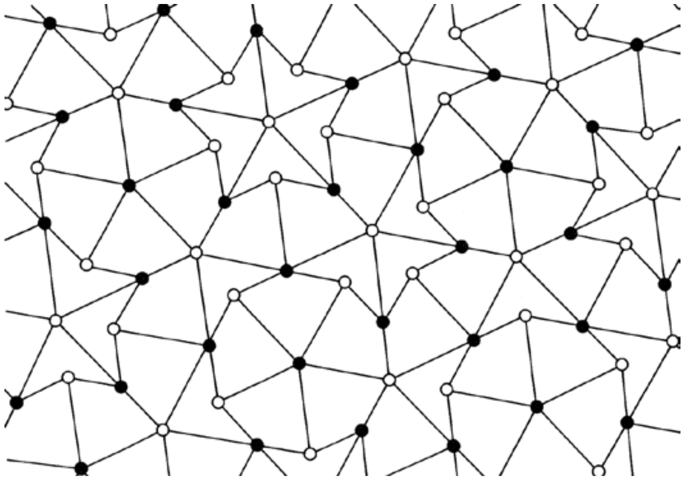


Figure 6.8 A Penrose tiling made with kite and dart tiles. Here the connection rules are marked with small black and white circles in the corners of the tiles, from Grünbaum & Shephard (1987).

Penrose managed to develop his system further, reducing the number of different tiles needed first from six to five and eventually to only two.[359] The mathematician John Conway named these two tiles as "kite" and "dart"; see Figs. 6.8 and 6.10. This dart and kite tiling can be transformed to another tiling with exactly the same properties but using two rhombuses instead.

The news about the discovery of this set of aperiodic tiles was apparently already circulating among mathematicians, and Richard K. Guy published a short article – with no images – about the unusual properties of the tiling in 1976.[360] Penrose had applied for patents for his tilings in Europe, the USA and Japan, and therefore a complete description with images was published only after the patents were secured. Martin Gardner (1914–2010), a prolific mathematical journalist, did this publishing in his article in January 1977 in the *Scientific American*.[361]

It is important to understand that the Penrose tiles contain certain connection rules which *force* nonperiodicity. All rhombuses, for example, are able to tile the plane periodically in a trivial way with the result looking like a compressed grid of squares: see, for example, Fig. 6.6. Such a repeating pattern is not met in the Penrose

[359] Penrose (1979)

[360] Richard K. Guy, "The Penrose Pieces", *Bulletin of the London Mathematical Society,* 8 (1976), pp. 9–10.

[361] Gardner (1977), pp. 110–121.

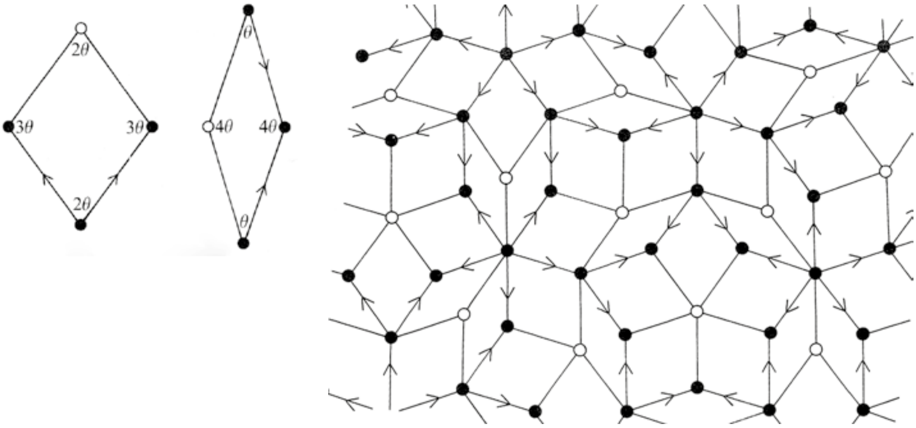


Figure 6.9 A Penrose tiling made with two rhombuses. The connection rules are marked to the edges with arrows (left), and a part of the tiling with the arrows depicted (right), from Grünbaum & Shephard (1987).

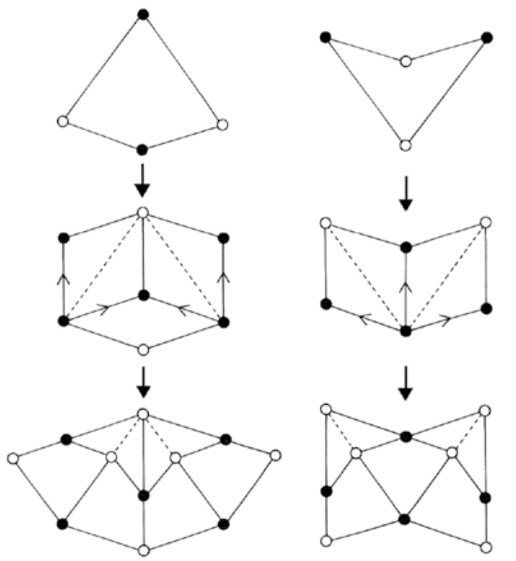


Figure 6.10 Kite and dart tiles inflated with rhombuses as an intermediate step, or rhombuses inflated with kite and dart as an intermediate step, from Grünbaum & Shephard (1987).

tilings. It is said that the tiles are *decorated* if there are rules restricting the way they can be connected. Another way of doing the restrictions is to manipulate the straight edges in such a manner that an edge only fits together with a corresponding edge. More common, however, is that some lines, arrows, or arches are drawn in the tiles (hence "decoration") in such a manner that the pattern has to continue smoothly from one tile to another, or that the corresponding markings on two neighbouring tiles have to match on the shared edge or on the shared vertices; see Figs. 6.8 and 6.9.

Another way of constructing a Penrose tiling is to use *inflation* – as the technique was more often called in the 1970s – or *substitution* as the technique is nowadays more often called in the technical literature.

In 1981 the Dutch mathematician Nicolaas Govert de Bruijn (1918–2012) published a third way of obtaining Penrose tilings. De Bruijn showed that all Penrose tilings are legitimate two-dimensional projections from a five-dimensional orthogonal lattice made of hyper-cubes.[362] The Penrose tiling pattern is not unique: its prototiles allow infinitely many different tiling patterns; thus, we can speak of Penrose tilings in the plural.

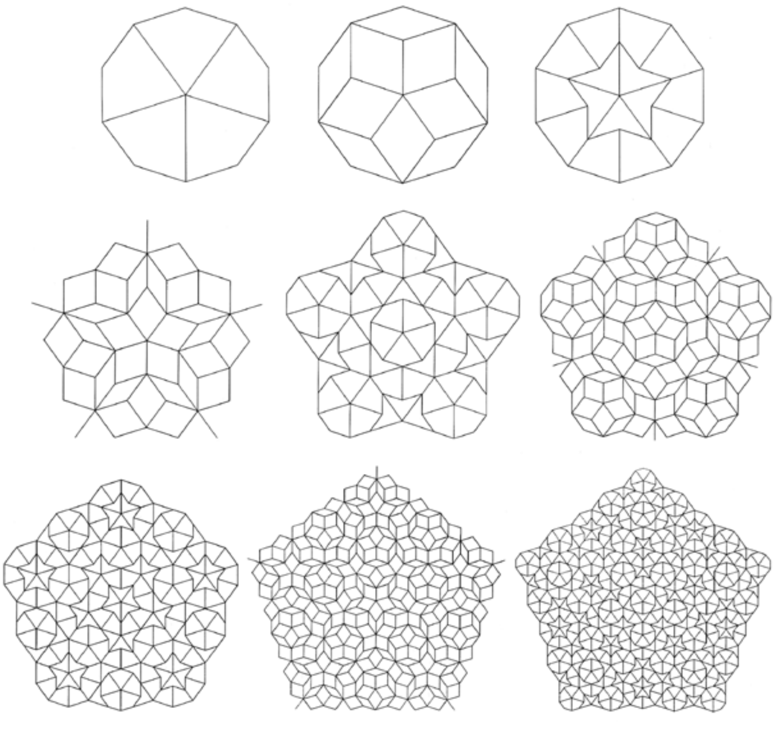


Figure 6.11 Eight steps in the inflation, or substitution, of one particular Penrose tiling. In reality, the pattern does not stay the same size, as in this illustration. The length of the edge of the tiles stays the same. In each step, the pattern grows and covers a larger area, and the limit of such a process is a complete tiling of the plane, from Grünbaum & Shephard (1987).

362 N. G. de Bruijn, "Algebraic Theory of Penrose's Non-periodic Tilings of the Plane, I–II", in *Proceedings of the Netherlands Academy of Science / Indagationes Mathematicae*, Vol. 84, Issue 1 (March 20, 1981), pp. 39–52 and 53–66.

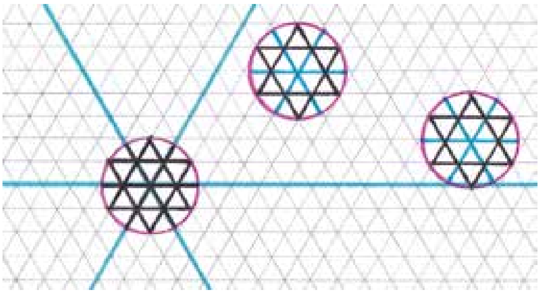


Figure 6.12 An example of sixfold rotationally symmetric periodic tiling made of equilateral triangles. Around the central point a patch is selected and two arbitrary copies of its pattern are highlighted (pink circles).

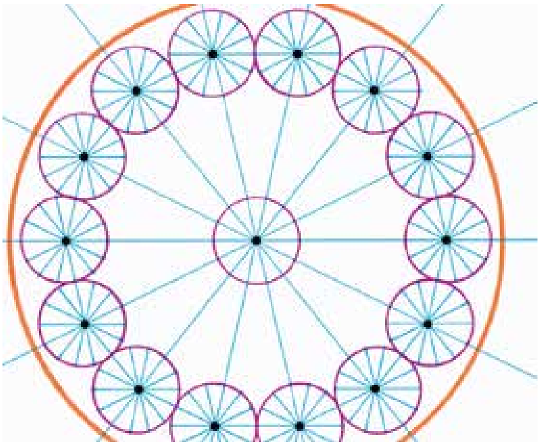


Figure 6.13 A schematic representation of a 14-fold rotationally symmetric pattern. Fourteen locally symmetric patches are marked around the centre of the global rotation. Note: no tiles are shown in this schematic image.

Quasiperiodicity

A tiling is *quasiperiodic* if for every finite pattern P of the tiling, there exists such a finite radius R that for every point X in the tiling, within a maximum distance R, a copy of the pattern P can be found.[363] A quasiperiodic tiling can have a global *n*-fold rotational symmetry also for values other than 2, 3, 4, or 6, which is not possible for a periodic tiling. In periodic tilings, only finite patterns with a local rotational symmetry for values other than 2, 3, 4, or 6 can be found. Thus, quasiperiodicity is a wider property than periodicity, including the latter. Every periodic tiling is quasiperiodic, but not vice versa.

If a periodic tiling has a rotational symmetry, the centre of rotation is not fixed in an absolute sense: any vertex can be considered as the centre of global rotation.

[363] In some definitions of quasiperiodicity, rotation is also allowed in order to consider the pattern P and its copy identical. In periodic tilings, a copy must be found using translation only (email correspondence with Jarkko Kari, 2017-03-31).

One such example is shown in Fig. 6.12, where the centre of the global sixfold rotational symmetry is marked with long blue lines meeting at the point. Two arbitrary copies of the pattern around the central point are selected and highlighted (pink circles) in the image. We can easily see that the tiling and its rotationally symmetric properties remain unaffected if we move the centre of global rotation (long blue lines) from its current location to any other location with a similar pattern. The same holds for all tilings depicted in Fig. 6.6.

In literature there are also other terms used for quasiperiodicity and related properties. Physicists seem to favour the expression *locally isomorphic*, which means that separate finite areas are similar, that is, isomorphic. The same expression is also sometimes used in mathematics with a similar meaning as for quasiperiodicity.[364] Another expression for quasiperiodicity is *uniformly recurrent*, which refers to the uniformity given by the fixed radius R that was used in the definition of quasiperiodicity a few pages ago. If the distance R is linearly dependent on the diameter d of the selected pattern P, the system is said to be *linearly recurrent*. Let me remark here that unfortunately, the terms aperiodic, nonperiodic, and quasiperiodic have been used in extremely mixed ways since the 1970s, and there is no end visible to this confusion. Some authors use the terms "aperiodic" and "quasiperiodic" in more relaxed sense of "nonperiodic", while some authors use the term "quasiperiodic" in the more strict sense of "aperiodic". Thus, great care is needed in checking what a particular author means with these particular terms. In this thesis, I use these terms to mean that aperiodicity is a much stronger condition than nonperiodicity, and to include periodicity in the category of quasiperiodicity.

The nonperiodic Penrose tilings with their fivefold symmetry are not also quasiperiodic, that is, uniformly recurrent. In other words, even if the Penrose tiling is not repeating in a periodic manner, we can take any patch of it, of any finite size, and always find identical copies of this patch within a limited distance from any point in the tiling. The maximum distance R we have to travel before finding an identical pattern is linearly dependent from the diameter of the selected patch; therefore the tiling is linearly recurrent by definition. But unlike one might first imagine, the required distance R is actually *very* short in the Penrose tilings: it is only twice the diameter d of the selected patch, that is: R(d) = 2d.[365]

[364] Grünbaum & Shephard (1987), pp. 558–568. Also Richard K. Guy used this phrase in his 1976 article: "One of the most remarkable theorems is the local isomorphism theorem (due to Penrose): given any (legal) portions of tiling of diameter d, then within distance R(d) of any given point in any other (sufficiently large portion of) tiling is an isomorphic copy of the given portion." Guy (1976), p. 10.

[365] This result is due to Conway; see Gardner (1977), pp. 116–117.

The vertex points of the Penrose tiles, as in any lattice point systems, must have a fixed minimum distance between them, meaning that points in different vertices cannot be arbitrarily close one another. From this follows that if the tiling has a point around which it is rotationally fivefold symmetric, there cannot be another similar point of rotation as this would lead to a contradiction.[366] Rotating the pattern around two such "centres" would create a situation where the minimum distance between some vertex points would become smaller than the fixed minimum distance. This means that there exists a unique centre in an absolute sense, around which the whole tiling can be turned and it still looks the same. All this may not sound strange or impressive at all, but let me present the matter in another way.

The Penrose tiling has a centre that is unique in some absolute sense. Take a patch P consisting of some tiles in the neighbourhood of the central point, then select an arbitrary point X1 from anywhere in the tiling and draw a circle with a radius R around that point. Then, inside this circle with a central point X1, a copy of the patch P can be found. How large can the patch P be? It can be as large as we want as long as R is twice the diameter of P. As we can select the point X1 arbitrarily from anywhere in the infinite tiling, there are also infinitely many copies of the "central" patch P in the tiling. How can the central point be unique in any absolute sense if there are infinitely many locations in the tiling having a perfectly identical pattern with the "central" patch P? Let me provide an explanation, or at least a clarification for this paradoxical-sounding situation: the centre defines the position of a pattern, but a pattern does not define the position of the centre.[367]

Against All the Odds: Quasicrystals

From 1981 to 1983 while on his two-year sabbatical, the Israeli chemist Dan Shechtman studied rapidly solidified aluminum metal alloys as a guest researcher at the U.S. National Bureau of Standards[368] in Washington, D.C.[369] On April 8th 1982 Shechtman made a surprising discovery from an x-ray diffraction image of crystals of one particular aluminum-manganese (Al-Mn) alloy: he noticed that the pattern in one diffraction image possessed *tenfold* symmetry.[370] This should not

366 See, for example, Buerger (1971), pp. 20–22, or Coxeter (1962), pp. 60–61.

367 My phrasing.

368 Since 1989 it has been National Institute of Standards and Technology (NIST).

369 http://www.nist.gov/public_affairs/releases/shechtman-100511.cfm (2015-11-30).

370 An image of Shechtman's laboratory logbook from April 8th 1982 is available at https://web.archive.org/web/20021205185209/http://www.quasi.iastate.edu/discovery.html with his words "10 fold???" (accessed 2015-11-30).

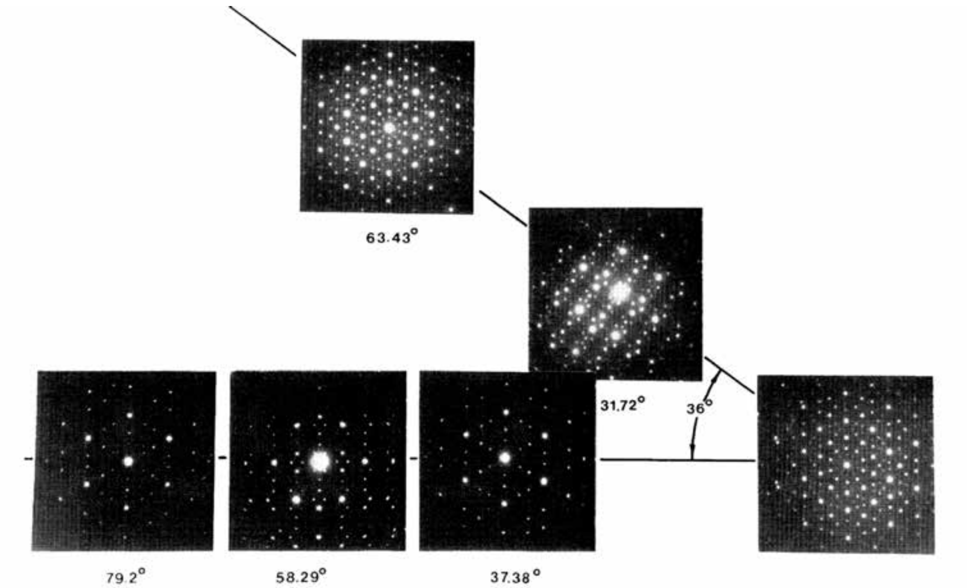


Figure 6.14 D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, (1984): diffraction patterns of an Al-Mn specimen with strange tenfold rotational symmetry (topmost and rightmost images). Small white dots *are not* images of individual atoms; they are condensed "wave fronts" diffracted from the ordered atomic structure of the crystal. If there were no ordered structures inside the specimen, no distinct dots would emerge in the diffraction image.

have been possible; no crystal of any known matter should have had a tenfold symmetry. As mentioned earlier, the crystallographic restriction theorem says that all possible rotational symmetries in crystals are either twofold, threefold, fourfold, or sixfold, but *never* fivefold, sevenfold, eightfold, ninefold, or tenfold, at least as long as the crystal is periodic, which was thought to always be the case.

It eventually took Shechtman and his three fellow scientists two and half years for the article announcing the finding to get published.[371] The discovery seemed to break most fundamental rules of crystallography, as it suggested that not all atoms are periodically ordered in solid crystalline matter. It should be mentioned here that the word "crystal" in material science means any piece of homogenous solid substance with a regularly ordered atom structure. This internal structure causes

371 D. Shechtman, I. Blech, D. Gratias, J. W. Cahn, "Metallic Phase with Long-Range Orientational Order and No Translational Symmetry", *Physical Review Letters,* Vol. 53, No. 20 (Nov. 12, 1984), pp. 1951–1953.

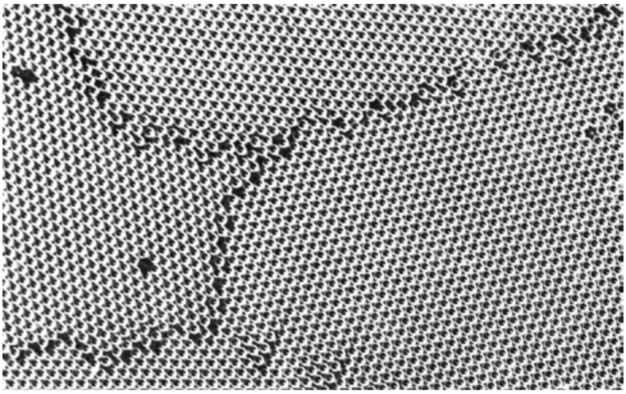


Figure 6.15 A mass of small uniform soap bubbles, serving as a model of the packing of atoms in polycrystalline matter; from Cyril S. Smith & John G. Burke, *Atoms, Blacksmiths, and Crystals*, (1967).

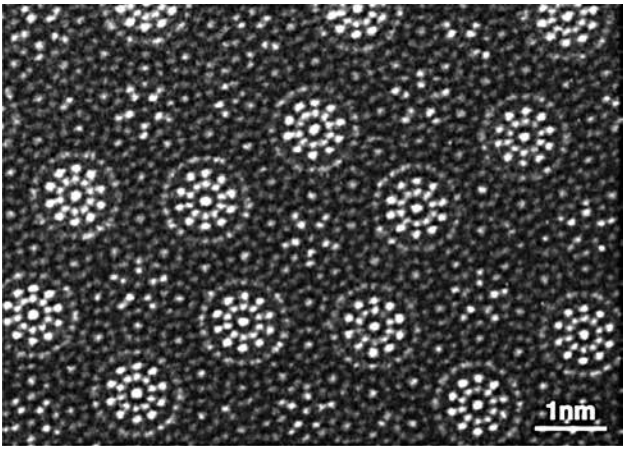


Figure 6.16 a scanning transmission electron microscope (STEM) image with ultrahigh resolution of a decagonal Al70-Mn17-Pd13 quasicrystal with local fivefold symmetries, from Eiji Abe (2012).

the crystal's external form with its geometric faces. There are also solid materials which *do not* possess any molecular-level periodic structure, such as glass. Thus, it is ironic that one such material is commonly called "crystal glass", which in a scientific sense is no crystal at all.

We can imagine that in crystals, atoms are packed next to each other in straight lines. In normal crystalline matter, such periodically ordered areas, or actually volumes in three dimensions, are placed next to each other but not necessarily with their internal atomic structure in the same orientation as their neighbours'. In the border areas of such otherwise homogenous matter, the pattern breaks and disorder emerges; see Fig. 6.15 above. Such matter is said to be *polycrystalline*, meaning that there are many internally ordered crystalline cells, or "grains" stuck together in a non-orderly manner. In addition, there are always some impurities and dislocations in real matter. In principle, when one sees a "perfect" crystal, let us say, even a few millimetres across, it is not only that the atoms inside it sit orderly in tidy rows, columns, and layers, but that there is only one such uninterrupted system throughout the whole specimen, continuing distances, which are measured in millions and millions of uniform atoms.[372]

What was then this novel material if it was not common amorphous glass producing no diffraction patterns or classical periodic crystals capable of producing diffraction patterns with only 2-, 3-, 4-, or 6-fold symmetry? It turned out that the structure observed by Shechtman with its rotational symmetry resembled very much the structure seen in the Penrose tilings. As it happened, the British crystallographer Alan L. Mackay (b. 1926) published two articles in 1981 and 1982 explicitly connecting the Penrose tilings and crystallography.[373] In 1981, one year before Shechtman made his discovery, Mackay wrote: "Here we will be confined to a partial discussion of one of the most striking absences from classical crystallography, namely that of pentagonal symmetry […] It gives an example of a pattern of the type which might well be encountered but which might go unrecognized if unexpected."[374]

Mackay's timing was almost prophetic, and luckily the encountered tenfold type – duplicating the fivefold symmetry – did not go unrecognized the next year, even if it was most unexpected. In fact, in 1962 Mackay published an article in which he proposed an icosahedral structure for a cluster of atoms, introducing the "forbidden" fivefold symmetry to crystallography.[375] Apparently, there was something "in the air" in the beginning of the 1980s in the field. The theoretical physicist and cosmologist Paul J. Steinhardt (b. 1952) in particular had done independent research and computer calculations about the possibility of an ideal icosahedral structure of atoms in 1981 and 1983.[376] When the discovery of the new class of "crystalline" solid matter with a quasiperiodic structure was published

[372] The distances between atoms in crystalline matter are of the order 10^{-10}m; see, for example, Buerger (1971), p. 13.

[373] Alan L. Mackay, "De Nive Quinquangula: On the Pentagonal Snowflake", *Krystallografiya*, Vol. 26 (1981), pp. 910–919, an English version: *Soviet Physics–Crystallography*, Vol. 26 (1981), pp. 517–522, and Alan L. Mackay, "Crystallography and the Penrose pattern", *Physica A*, Vol. 114A, No. 1–3 (1982), pp. 609–613. Mackay's title "De Nive Quinquangula" is of course a jocular reference to Kepler's *De Nive Hexangular* (1611).

[374] Mackay (1981), pages 517 & 522 in the English version.

[375] Alan L. Mackay, "A Dense Non-Crystallographic Packing of Equal Spheres", *Acta Crystallographica*, Vol. 15 (1962), pp. 916–918.

[376] Paul J. Steinhardt, David R. Nelson, and Marco Ronchetti, "Icosahedral bond orientational order in supercooled liquids", *Physical Review Letters*, Vol. 47, pp. 1297–1300, (1981), and Paul J. Steinhardt, David R. Nelson, and Marco Ronchetti, "Bond-orientational order in liquids and glasses", *Physical Review B,* Vol. 28, p. 784 (1983).

in 1984, Steinhardt and physicist Don Levine quickly named it "quasicrystals".[377] The report of the existence of such strange solid matter was not universally met with applause. The two-time Nobel laureate, chemist and one of the most famous scientists of his time, Linus Pauling (1901–1994), adamantly and publicly opposed the very idea of quasicrystals until the end of his days.[378] He is even reputed to have said: "There are no such things as quasicrystals, only quasi-scientists." [379]

Three-dimensional icosahedral structures have axes of twofold, threefold and fivefold rotational symmetries.[380] Icosahedral quasicrystals are the most common, but a few other types of quasicrystals have also been found.[381] These other types are named after the rotational symmetries detected in them, that is, octagonal (8-fold), decagonal (10-fold), and dodecagonal (12-fold) types of quasicrystals.[382] Colloidal water-based quasicrystals with 12- and 18-fold symmetries were discovered in 2010.[383] So far, no quasicrystals with 7-fold, 9-fold, or 11-fold symmetries have been observed, but even if they may seem unlikely to exist, such types are perhaps not altogether impossible.[384]

After the discovery of quasicrystals, the classic definition of crystal as a periodically repeating regular arrangement of atoms no longer applied, and eventually the very definition of "crystal" had to be redefined in crystallography. Since 1992 the International Union of Crystallography (IUCr) has defined "crystal" and "aperiodic crystal" in the following way: "by *crystal* we mean any solid having an essentially

[377] Don Levine and Paul J. Steinhardt, "Quasicrystals: A New Class of Ordered Structures", *Physical Review Letters,* Vol. 53, No. 26, pp. 2477–2480; Dec. 24, 1984.

[378] Linus Pauling, "Apparent Icosahedral Symmetry is due to Directed Multiple Twinning of Cubic Crystals", *Nature*, Vol. 317 (1985), pp. 512–514, and Linus Pauling, "So-called Icosahedral and Decagonal Quasicrystals are Twins of an 820-atom Cubic Crystal", *Physical Review Letters*, Vol. 58 (1987), pp. 365–368. Both papers by Pauling are reprinted in Steinhardt & Ostlund, *The Physics of Quasicrystals* (1987), pp. 322–331.

[379] http://www.nature.com/milestones/milecrystal/full/milecrystal20.html (read 2015-12-01).

[380] All these symmetries are clearly shown in Shechtman *et al*. (1984); see Fig. 6.14.

[381] See, for example, Paul J. Steinhardt and Stellan Ostlund, *The Physics of Quasicrystals*, 1987, or Christian Janot, *Quasicrystals; A Primer*, 1994, or see the following footnote.

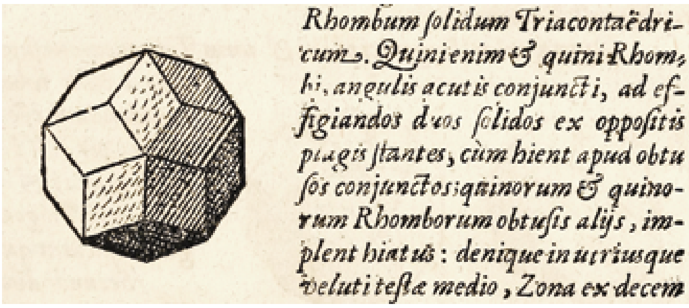
[382] Walter Steuer and Sofia Deloudi, "Fascinating Quasicrystals", *Acta Crystallographica Section A*, Vol. A64 (2008), pp. 1–11. This concise feature article is also available online at http://www.math.uiuc.edu/~gfrancis/quasicrystals/public/IonBaianu/steurer.pdf (last accessed 2016-08-28). A more extensive mathematical analysis can be found, for example, in Joshua E. Socolar, "Simple Octagonal and Dodecagonal Quasicrystals", *Physical Review B*, Vol. 39, No. 15 (May 15, 1989), pp. 10519–10551, a publication with impressively large page numbers.

[383] Fischer *et al*., http://www.pnas.org/content/108/5/1810.abstract (accessed 2015-12-10).

[384] Steuer and Deloudi (2008), p. 3.

discrete diffraction diagram and by *aperiodic crystal*[385] we mean any crystal in which three-dimensional lattice periodicity can be considered to be absent." [386] In December 2011 Dan Shechtman was awarded the Nobel Prize in chemistry "for the discovery of quasicrystals." [387] Shechtman's prize was announced in September 2011, awakening my interest in rhombic tilings with non-crystallographic rotational symmetries.

I had been studying rhombic tilings on my own as a hobby every now and then for about 20 years before I entered the program for a doctorate in fine arts. The tiling system I had been irregularly working on was aimed to contain an infinite number of circular patterns with local *n*-fold rotational symmetries, as seen in Figs



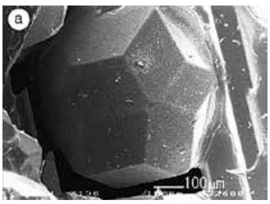


Figure 6.17 A rhombic triacontahedron (a solid with 30 rhombic faces) depicted by Kepler in *Harmonices Mundi* (1619), p. 61 (above), and a scanning electron microscope image of Zn-Mg-Sc quasicrystal from Ishimasa *et al.*, (2004), Fig. 3a (left).

385 An interesting coincidence is that the theoretical physicist Erwin Schrödinger (1887–1961) used the word "aperiodic" and the phrase "aperiodic crystal" in his 1944 book *What is Life?* Schrödinger wrote: "[T]he most essential part of a living cell – the chromosome fibre – may suitably be called an *aperiodic crystal*. In physics we have dealt hitherto only with *periodic crystals*. […] Yet, compared with the aperiodic crystal, they are rather plain and dull. The difference in structure is of the same kind as that between an ordinary wallpaper in which the same pattern is repeated again and again in regular periodicity and a masterpiece of embroidery, say a Raphael tapestry, which shows no dull repetition, but an elaborate, coherent, meaningful design traced by the great master." p. 3; italics by Schrödinger.

386 *Acta Crystallographica*, A48 (1992), pp. 928; these definitions are also available online at http://reference.iucr.org/dictionary/Aperiodic_crystal and http://reference.iucr.org/dictionary/Crystal (both accessed 2016-08-29)

387 http://www.nobelprize.org/nobel_prizes/chemistry/laureates/2011/press.html (2015-12-01)

6.11 and 7.1 for $n = 5$ and $n = 7$, respectively. I have included a description of this system in this thesis mainly for two reasons: firstly, the tiling system was completely solved only during my doctorate studies; that is, a mathematical verification of the general principle and numeric tables for a complete solution were obtained only in late 2011, and secondly, the first tiling system is an immediate predecessor of the second, mathematically more ambitious tiling system, which is more closely related to the theme of this thesis. In late 2011, I immersed myself with fresh energy in the study of tilings with non-crystallographic rotational symmetries. After some dead ends, a solution developed, fully utilizing the inflation, or "substitution", as also the Penrose tilings do. The solution enables quasiperiodic tilings with n-fold rotational symmetries for all n, all non-crystallographic rotational symmetries included as well. No such system had been constructed – or proved to exist – before 2011. In the following chapter, I will tell how this discovery emerged.

7 | Tiling from Seven to Eleven

In this chapter I describe how my own tiling system gradually developed during my studies for the doctorate in fine arts. I also provide the necessary information for understanding its geometrical logic.

> *These results require the use of ruler and drafting pen and establish unmodulated line as a legitimate artistic means. In this way they oppose a belief that the handmade is better than the machine-made, or that mechanical construction is anti-graphic or unable to arouse emotion.*
>
> Josef Albers, *Despite Straight Lines* (1961)

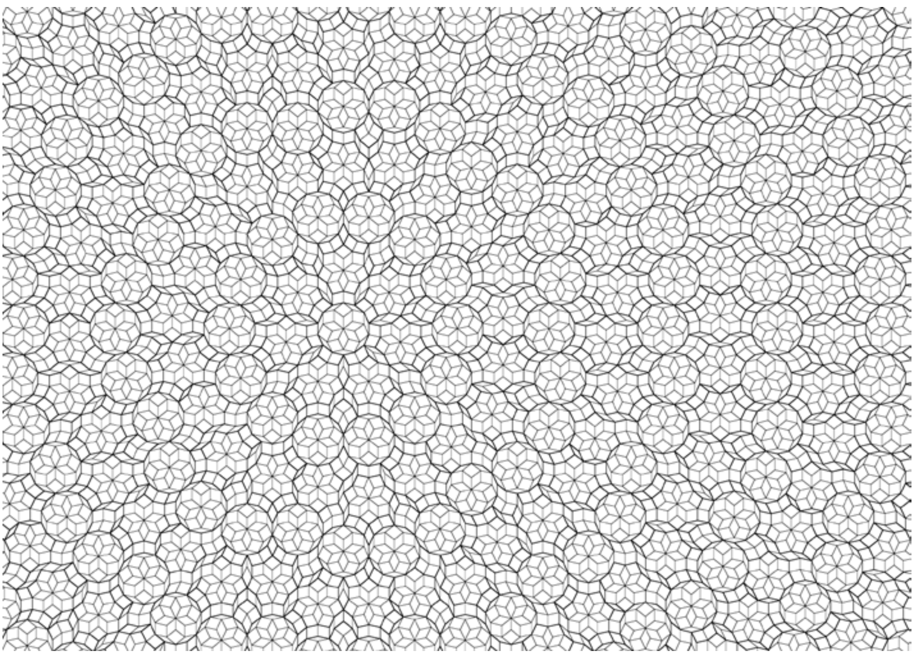


Figure 7.1 Markus Rissanen, Hex Rosa tiling for $n = 7$

Hex Rosa

I had studied rhombic tilings with non-crystallographic rotation symmetries quite a long time before my doctoral studies. Nevertheless, a complete solution for my "first tiling system" was reached only at the end of 2011. Although the actual tiling consists of rhombuses, this first system is based on using provisional *hexagons* and *rose*-like patterns, hence the name *Hex Rosa*. In the following, I give only a short description of the main lines of this system. A more comprehensive description of this system[388] is provided in this thesis as Appendix A (Hex Rosa).

The key idea behind Hex Rosa tilings is my simple observation that there is a certain hexagon which has fairly interesting properties when seen from the perspective of rotational symmetry. I call it *delta hexagon*. This hexagon has all sides of equal length and opposite edges parallel. Every integer $k > 1$ unambiguously defines one such hexagon with interior angles of 360°/ k in two opposite corners and 180°((k–1)/ k) in four other corners, the sum of which is always 720°. Delta hexagons can be arranged edge-to-edge around a central point in such a manner that the resulting tiling is k-fold rotationally symmetric, for example, for $k = 2$ and $k = 3$, as depicted in Fig. 7.2 (left) and (right), respectively. Fig. 7.2 also shows how delta-hexagons are arranged in V-shaped "wedges" that meet in the centre, or, more precisely: k wedges meet at the central point, and the k wedges between them do not meet in the centre. Note that rectangles seen in Fig. 7.2 (left) are also hexagons with interior angles of 90°, 90°, 180°, 90°, 90°, and 180°. Fig. 7.3 shows the same

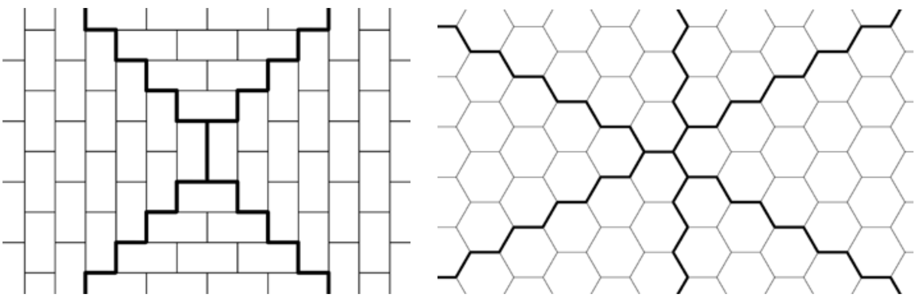


Figure 7.2 Provisional superstructures used in Hex Rosa tilings. Specific delta hexagons can tile the plane by forming never-ending "wedges" which grow away from the centre. The tiling at left is 2-fold, and the tiling at right, 3-fold rotationally symmetric with 4 and 6 "wedges", respectively.

[388] Rissanen, "Hex Rosa", in *Bridges Finland Conference Proceedings*, 2016, pp. 209–216. The paper is available online at http://archive.bridgesmathart.org/2016/bridges2016-209.html (accessed 2016-07-07).

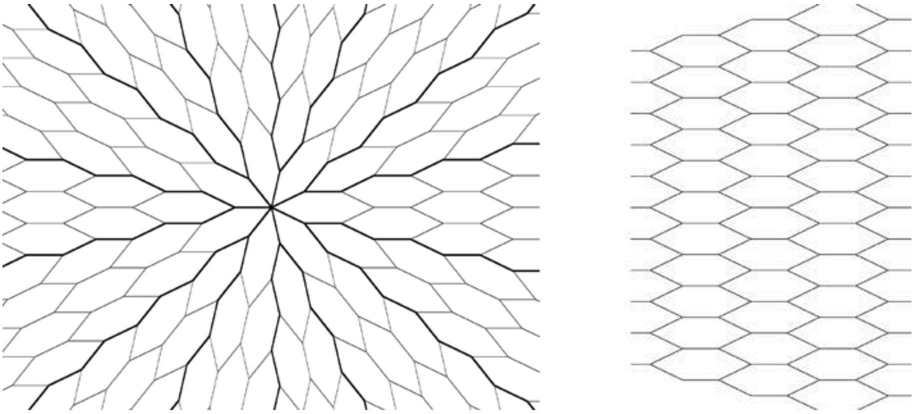


Figure 7.3 A 7-fold rotationally symmetric tiling made of “wedges” (left) consisting in turn of delta hexagons. However, removed from the centre, the tiling inside a “wedge” is no different from a monotonous “chicken wire” pattern made of hexagons (right). Compare with Fig. 6.4 (bottom right).

configuration for $k = 7$; compare this with Fig. 6.4 (bottom right). The necessity of using two separate letters, k or n, is discussed in Appendix A. In Hex Rosa tilings, for all odd n, the simple relation $k = n$ holds, whereas $k = n / 2$ for all even n.

As mentioned, the delta hexagon (and also the wedges it can collectively constitute) may be considered as only a provisional structure for producing rhombic tilings with global and local rotational symmetries. Fig. 7.5 depicts how in a Hex Rosa tiling, here, for example, for $n = 11$, small rhombuses are placed inside the relatively large delta hexagon. Note how there is a rose-pattern made of rhombuses in every corner of this hexagon. Fig. 7.7 depicts such n-fold rotationally symmetric “roses” for n = 3, 4, 5, 6, 7, and 8. The real challenge in this system was not how to construct the delta hexagon for arbitrary n – which is a trivial task – but how to fill the interior of the large hexagon with tiny rhombuses in such a manner that there are infinitely many locally rotationally symmetric “roses” equally distributed along the whole finished tiling. One result of such a pattern is seen in the beginning of this Chapter; see Fig. 7.1.

During my doctoral studies I managed to construct a second, mathematically further-reaching system which was based on my first system. This second system also used “roses”, but was not based on using provisional hexagons; rather, it used a more sophisticated geometric method called *substitution*, hence the name *Sub Rosa*. A co-authored article with a mathematically valid proof of the main result of this second system was published in June 2016.[389] Due to the rather technical nature of the paper, it is provided in this thesis as Appendix B (Sub Rosa). In this Chapter, I have considered it suitable to give a less formal description of this system using less technical language than is customary in the mathematical literature. As the substitution operation plays a prominent role in the following section, I will continue by explaining it first.

Substitution

Substitution is a two-part process which generates a larger pattern from a smaller pattern in the following manner:

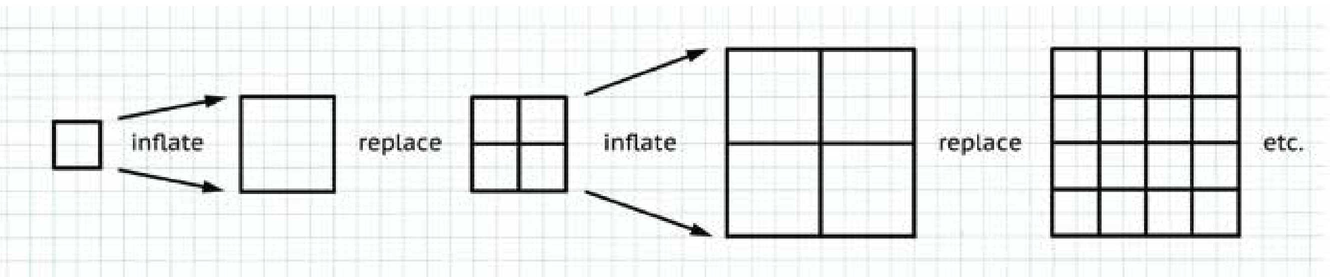


Figure 7.4 A substitution of a square with four squares; the process is repeated here twice, producing 4^2 = 16 squares of the original size. The third step would produce 4^3 = 64 squares.

Fig. 7.4 depicts a very simple substitution. Let us assume that the length of the edge of the small square is one unit; hence we call it "unit square". The unit square is enlarged, or "inflated", or "scaled" with a scaling factor S, which, in this example, equals two: S = 2. The inflated square is dissected, or divided into, replaced, or *substituted* with four smaller squares that are the same size as the original unit square. As a result of this operation, one unit square transforms into four unit squares. The process may be repeated, which transforms four unit squares into sixteen unit squares, and the next step transforms them into sixty-four unit squares, etc. The scaling factor S does not have to be two, however; in

[389] Jarkko Kari and Markus Rissanen, "Sub Rosa, a System of Quasiperiodic Rhombic Substitution Tilings with *n*-Fold Rotational Symmetry", *Discrete & Computational Geometry*, Vol. 55, Issue 4 (June 2016), pp. 972-996, published first online 2016-04-04 at http://link.springer.com/article/10.1007/s00454-016-9779-1 a free (pre-review) version of the paper is also available online at http://arxiv.org/abs/1512.01402 (since 2015-12-04).

this particular example, it could be any integer larger than one, for example, three, seventeen, or one million.

The scaling factor has to be such that the units of the inflated and dissected patch are congruent with the previous units; in the case of the square, this limits the scaling factors to integers, but this is not the case in substitution tiles in general. Nor does the shape have to be a square or a parallelogram. Every triangle, for example, can be divided into four, nine, sixteen, twenty-five, etc. triangles which are congruent with the original triangle.[390] The resulting patch grows fast in the process, exponentially, to be exact. Thus, only a proper scaling factor and replacement rule are needed to tile an infinite plane; actually, an adequate replacement rule is enough, as it also contains the measure of the scaling factor in pictorial form. Thus, all the necessary information for a substitution process concerning one tile can be presented with only one image containing the original tile and its larger counterpart after one substitution step. The presentation in Fig. 7.4 is superfluous; only one 4x4 square is needed as it shows how inflated the original square is and in which way the inflated square is divided.

Substitution "inwards"

Just as we can inflate and replace the unit-square with a larger-sized square containing a substructure made of congruent copies of the original unit-square, we may also approach the same process but in the opposite direction. In the case of a square, we replace the original unit-square with four smaller squares, each having an edge of 1/4 of the unit of the original length. These subunits are again replaced in the same way with even smaller squares, that is, squares with an edge of 1/16 units, etc. These two processes, substituting "outwards" and substituting "inwards", are like zooming in or out, respectively, an infinitely repeating structure which manifests the same pattern at every level of magnification. If a substitution can be done at all, it can be done in both directions.

Let me remark that in mathematics, substituting "outwards", that is, the covering of an infinite plane with unit-tiles, is the only direction sought after; the unit-tiles are not considered to be divided into smaller and smaller subunits. In a proper tiling, a fixed limit is required for the size of all tiles: infinitesimally small tiles are not allowed. Nonetheless, while conducting this study of tiling systems (2011–2012), I still believed that the unit forms should also be divided into smaller and smaller subunits, even if such are not explicitly depicted in the image. I thought that all those infinitely small tiles somehow existed "down there", either actually or potentially, even if we don't have any use for them, or even if we

[390] I leave it to the reader to figure out how this dissecting is done for a triangle. The solution is trivial and Fig. 7.4 serves as a good example, *mutatis mutandis*.

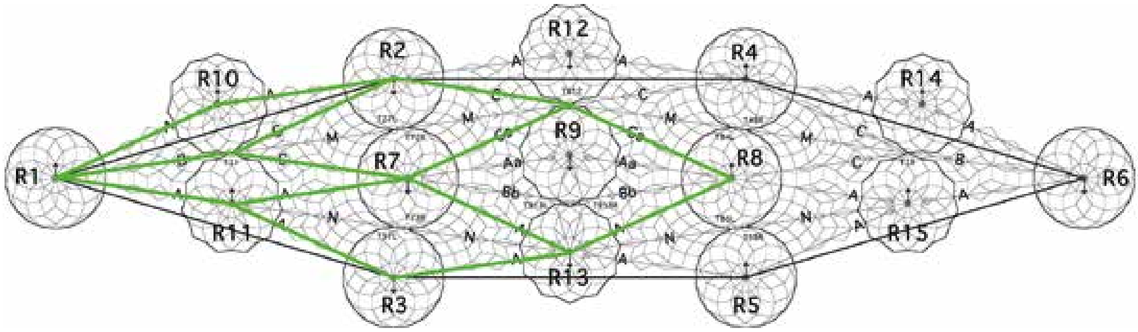


Figure 7.5 The first idea of a substitution tiling based on the Hex Rosa tiling system.

don't pay any attention to them. I believe this erroneous way of thinking actually helped me to have the crucial first idea(s) of my tiling system, which I describe next. Let me remark that all examples in the following are only for odd values as I had concentrated nearly all of my efforts on those and because the system for even values differs slightly from odd cases.[391] Note also that all rhombuses are connected edge-to-edge.

Towards a Substitution

As stated in Appendix A, the configuration depicted in Fig. 7.5 gave me the first idea of transforming a Hex Rosa tiling into a new system based on the substitution process. As in the beginning of this Chapter, in the following, we also assume the lengths of the edges of all small rhombuses to be one unit; hence I will call them *unit rhombuses*. A generic Hex Rosa contains specific essential straight lines which also define larger rhombuses, identical in form but not in size to the unit rhombuses. The edges of these larger rhombuses are shown in green in Fig. 7.5.

Seeing these larger rhombuses in a sketch similar to Fig. 7.5 immediately raised in my mind the question: if unit rhombuses can be arranged to form such larger "super-rhombuses", would it not be possible also arrange these larger "super-rhombuses" to even larger "super-super-rhombuses"? Or, if this was possible, would it not mean that all unit rhombuses could be considered as consisting of smaller "micro-rhombuses" and so on, *ad infinitum*, in both directions. At this point, it occurred to me that the super-rhombuses depicted in green in Fig. 7.5 are

[391] For the solution for even values, see Appendix B, Part 4 "Substitution Rules for Even *n*."

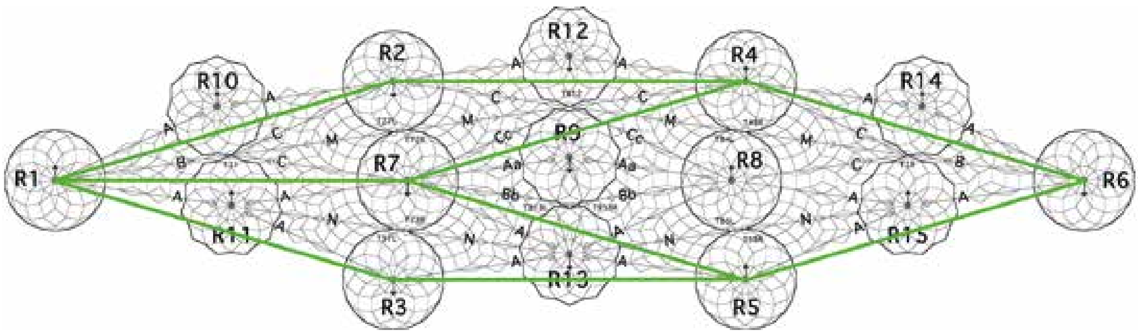


Figure 7.6 The second idea of a substitution tiling based on the Hex Rosa tiling system.

not perhaps the most natural candidates for such a super-structure as the essential hexagonal shape R1–R2–R4–R6–R5–R3 itself can be divided easily into two thin rhombuses: R1–R2–R4–R7 and R1–R7–R5–R3, plus one slightly heftier rhombus R7–R4–R6–R5, all depicted in green in Fig. 7.6.

This second idea seemed more promising at first glance as the super-rhombuses depicted in Fig. 7.6 seemed to connect with each other more easily than the super-rhombuses depicted in Fig. 7.5. To help the readers who have not yet read the Appendices, it may be mentioned here that the circular configuration, *rose*, essential to both of my systems, is located in every corner of the hexagonal form in Figs. 7.5 and 7.6.[392] The roses for $n = 3, 4, 5, 6, 7$, and 8 are depicted in Fig. 7.7. It seemed a good idea to have a rose at each corner of every super-rhombus. The roses located in the corners can be compared to pieces of a round cake, albeit with variable central angles not so often met in real cakes. If such sectors are recombined around a point, a complete rose of 360° is attained, no matter in which order they are put together. But as correct as this reasoning was, the situation turned out to be more complicated in the bigger picture.

Alternative Edges

The super-rhombuses depicted in Fig. 7.6 had a serious problem; no matter how their edges were chosen among unit rhombuses, they fit together only in a very limited way, and if a super-rhombus is rotated 180°, for example, its edge no longer connects with the neighbouring super-rhombus without the forbidden overlapping

392 In addition to the six corners of the hexagon, the rose pattern also resides in a few other points inside the hexagon and near the middle points of its edges. These roses are numbered from R1 to R15 in Figs. 7.5 and 7.6

or gaps. The situation depicted in Fig. 7.5 first seemed to have an advantage when compared to that depicted in Fig. 7.6.

In Fig. 7.5, every edge of each super-rhombus bisects all unit rhombuses on it.[393] This gives an excellent way of reckoning the unit rhombuses along the edge as they are uniformly orientated and fixed on a straight line like pieces of meat or vegetables on a skewer, unlike the unit rhombuses depicted in Fig. 7.6, where they jump up and down around the straight (green) lines that define the smooth “theoretical” edges of the super-rhombuses.

Let it be mentioned here that the edge of the super-rhombus does not need to run along a straight line; quite the opposite. It suffices that if a “cape” protrudes from the edge, a corresponding “bay” with the same shape needs to exist in the correct place to compensate. The shape of the edge of a super-rhombus can vary, but its *area* needs to be exactly equal to the area defined by the green lines in Figs. 7.5 and 7.6.

For a while it also seemed possible to re-order the rhombuses depicted in Fig. 7.5 along the edges to get super-rhombuses with all four edges having identical structures made of bisected unit rhombuses. Alas, this idea ran into trouble. The edges did not possess a symmetrical shape around their middle-points, meaning, for example, that edge R1–R11 matches with, for example, edge R1–R10, but not with edge R10–R2; the order of the unit rhombuses along edge R1–R11 have to be reversed to run not from R10 to R2 but from R2 to R10; see Fig. 7.5.

A Vision of a Tiling

It seemed that if there existed a solution for turning a Hex Rosa tiling into a substitution tiling, it should combine the best characteristics of the ideas depicted in Figs. 7.5 and 7.6, but somehow simultaneously to avoid all of their disadvantages. How this was to be achieved, I did not know, but I had a feeling that somehow it *might* be done. What drove me at the time was a vision of a tiling where combined unit rhombuses would constitute super-structures (super-rhombuses), which when combined would constitute super-super-structures, etc. At the same time all of the unit rhombuses would contain smaller and smaller “invisible” self-similar sub-structures, which might be visible if we could just “zoom in” far enough into unit rhombuses. In addition to just having such self-similar structures in both directions *per se*, such a tiling would also possess a global *n*-fold rotational symmetry, which I found interesting because such a property was also found in the Penrose tilings.

In addition, there would not only be global *n*-fold rotational symmetry but also an infinite number of small patches, the roses, with a spatially-limited local *n*-fold

[393] With lines Bb and Cc being the only exceptions for not bisecting their last rhombus.

rotational symmetry. I also reasoned that by placing super-rhombuses in the shape of a rose, something very interesting would happen. Such a "super-rose" would work as a perfect model for the whole tiling, rendering all other super-rhombus combinations simply *unnecessary*. The idea was that if the starting pattern were a rose inflating to a super-super-etc.–rose, at the same time there would be similar rose patterns in each corner of every super-rhombus, with smaller and smaller self-similar patterns inside each of them. This would mean that even the tiniest detail of the tiling would contain a perfect copy of the "whole" tiling. This would be analogous with the Koch snowflake or other self-similar fractals, where a small detail contains a complete copy of the "whole" object, with the difference that this kind of tiling system would also include the aspect of global *n*-fold rotational symmetry.

A Problem Arises

For the rest of 2011 I tried to get a substitution system working based on modified Hex Rosa tilings, but no matter how I tried, even the simplest case *n* = 5 with only two rhombuses (1,4) and (2,3)[394] seemed unsolvable; I just could not construct the super-rhombuses S(1,4) and S(2,3) with legitimate edges and corners with rotationally symmetric *n*-fold "type 1" roses.

The crux of the matter was that the sharpest corner of the super-rhombus S(1,4) simply could not be tiled with an *n*-fold rotationally symmetric rose pattern in the vertex. As bad as it was, the problem turned out to be not of a practical or technical nature; it turned out to be a logical impossibility. By definition, all my rose-patterns with *n*-fold rotational symmetry had centres divided into *n* parts, producing *n*

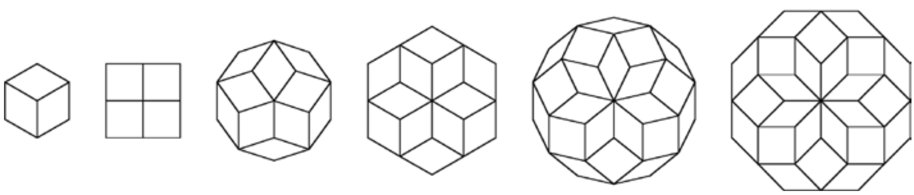


Figure 7.7 The type 1 rose for *n* = 3, 4, 5, 6, 7, and 8. The perimeter is convex, regular polygon. This pattern is *n*-fold rotationally symmetric.

394 These notations are explained in more detail in the Appendices, but I may as well explain here that (1,4) and (2,3) denote in the case of *n* = 5 rhombuses with the vertex angles of (36° and 144°) and (72° and 108°), that is, (1x 180°/5, 4x 180°/5), and (2x 180°/5, 3x 180°/5), respectively. The notation S(1,4) refers to a super-rhombus (hence "S") with the vertex angles of (36° and 144°).

number of unit rhombuses with an internal vertex angle of 360°/*n* meeting in the centre. On the other hand, my system was based on the assumption that there was a rose pattern with *n*-fold rotational symmetry located in every corner of the hexagonal form, and thus also in every corner of the super-rhombus I was trying to construct. However, as the thinnest super-rhombus S(1, *n*–1) had 180°/*n* as its smallest internal corner angle, it was simply too small an angle to contain the two-times larger angles of the unit rhombuses meeting (by definition) at the middle point of a rose. In the case of *n* = 5, for example, this meant that there was no way of placing a unit rhombus with a vertex angle of 72° inside a super-rhombus with a vertex angle of 36°. After realizing this insurmountable obstacle, not just with super-rhombus S(1,4) for *n* = 5, but with all super-rhombuses of type S(1, *n*–1) for all *n*, I put the whole thing aside, as I had no tools to tackle the problem any further.

The Solution

Thus the study of this problem stopped, or at least slid into a dormant state for about two or three months. But, although in a way I had given up the hope of finding a solution, the problem of *n*-fold rotationally symmetric substitutions

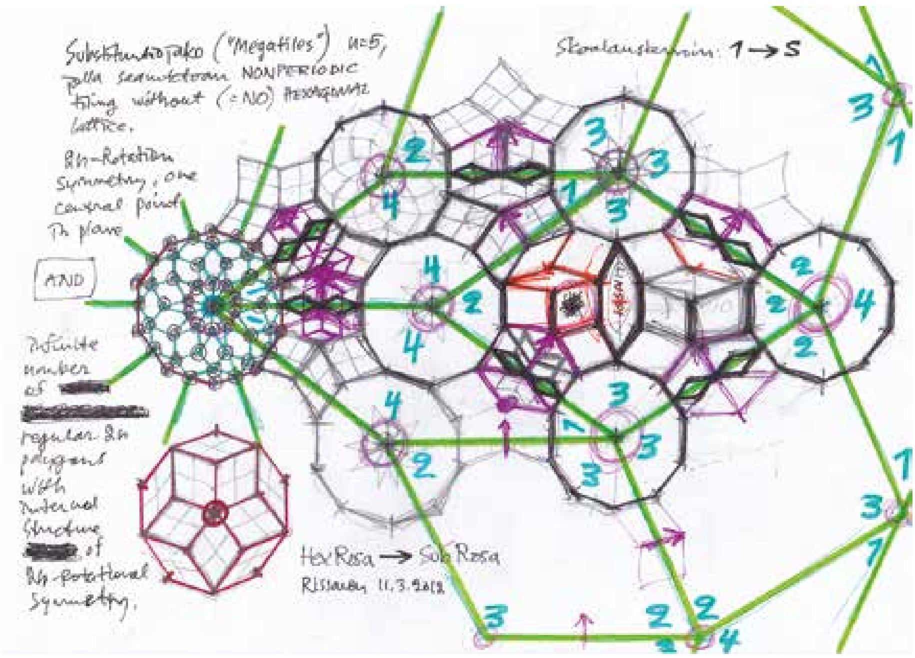


Figure 7.8 The first sketch of a successful substitution tiling dated 2012-03-11. The blue numbers express the angles (36°, 144°) and (72°, 108°) of the rhombuses (1,4) and (2,3), where the unit is 180°/5 = 36°.

did not leave me at rest. Then, late one evening in March 2012, an idea for circumventing this seemingly insurmountable problem popped into my head: If the problem is impossible to solve using *n*-fold roses, why not try 2*n*-fold roses instead? And, paradoxical as it sounds, the following worked: if you can't split your problem in half, double it.

Fig. 7.8 reproduces the first sketch of a solution. All rhombuses of the "old" *n*-fold rose could be divided into 2x2 smaller rhombuses, which can be rearranged *inside the perimeter* of the rose into a new, "second type" rose having 2*n*-fold rotational symmetry instead of *n*-fold rotational symmetry. In Fig. 7.8, the first "type 1" rose is depicted in red, left, with its 10 red-lined rhombuses divided into 2x2 smaller rhombuses, which in turn are rearranged into the second "type 2" rose, depicted in blue directly above the red one, consisting of 40 even smaller rhombuses. All of the unit rhombuses in a Hex Rosa tiling can also be replaced with four smaller rhombuses, which are then considered as new unit rhombuses with two times smaller edges. After this operation, these new unit rhombuses can be rearranged, and, for example, the "provisional" delta hexagon can now be divided into two S(1,4) super-rhombuses plus one S(2,3) super-rhombus in such a manner that all new unit rhombuses along the "theoretical" edge of a super-rhombus (depicted in green) are finally bisected by this very edge; see Fig. 7.8.

All of this combined opened up a way for a legitimate rhombic substitution tiling with a global 2*n*-fold rotational symmetry for all *n*, albeit without rigorous proof at the time. The system was based on the idea of having an infinite number of small patches, i.e. roses, each containing a perfect copy of the "whole" tiling. For example, in Fig. 7.8, zoom-level zero (S0) blue rhombuses of the "centre" rose (middle, left) are inflated into larger, zoom-level one (S1) green super-rhombuses, which share the same centre and configuration, which are in turn inflated into next-zoom-level two (S2) super-super-rhombuses, etc. This process can be done in both directions: zooming in and zooming out with the same structure repeating again and again, as is the case with fractals. From March 2012 onwards, I had the right implements to start constructing the actual substitution tilings.

The First Version

For each *n*, there is the same number of different unit rhombuses and super-rhombuses, both with identical angles, by definition. In all aspects, they are perfectly identical, save the size. For every *n*, their number equals the integer part of *n*/2. For example, for *n* = 9, there are four types of unit rhombuses: (1,8), (2,7), (3,6), and (4,5), and four types of super-rhombuses: S(1,8), S(2,7), S(3,6), and S(4,5), with exactly the same vertex angles but in a larger size. Only one super-rhombus of each type needs to be solved as the rest of the tiling is made of similar

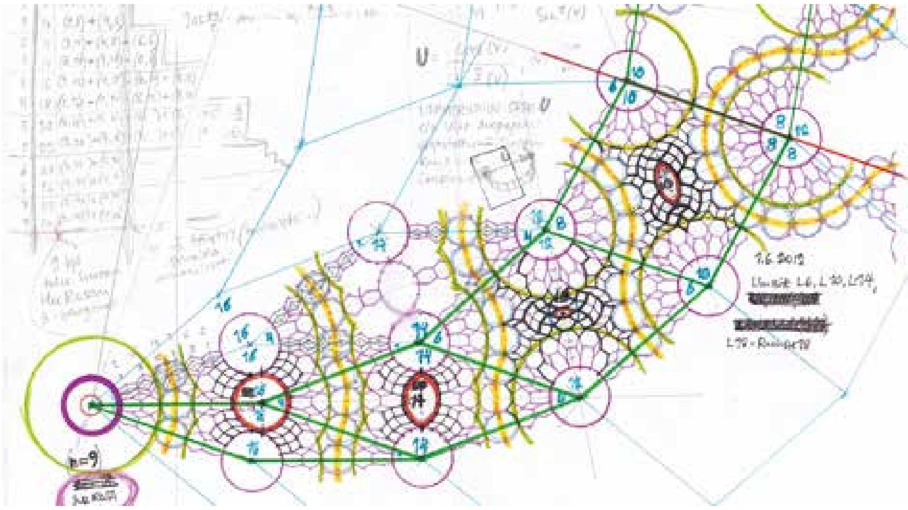


Figure 7.9 The first version of the Sub Rosa tiling for n = 9, sketch dated 2012-06-01.

super-rhombuses. Actually, only a quadrant of each super-rhombus would suffice as super-rhombuses are mirror symmetric along their diagonals, always meeting at right angles at the middle point of the rhombus. These diagonals are depicted with orange-red lines in Fig 7.11.

The first version of the Sub Rosa tiling was mathematically functional but aesthetically not that satisfying. The first system was directly based on Hex Rosa tilings, in which all unit rhombuses were first replaced with a 2x2 "window-pattern" of new smaller unit rhombuses, which could then be rearranged to enable a logical substitution. After mentioning this aesthetic aspect, I will not exhibit more tilings from the first version of the system. If a tiling was drawn into a fairly large scale and the tiny unit rhombus "windows" were left undrawn, the aesthetic result was not always unsatisfying, see, for example Fig. 7.9. Note, however, that in this particular image, all small black rhombuses are to be divided into 2x2 smaller rhombuses presenting the new unit rhombuses, as all purple hexagons are to be divided into 12 unit rhombuses, etc.

At this point in the Sub Rosa tilings, an interesting structure emerged in the shape of a continuous string of elliptical shapes. These elliptical shapes were positioned on arches which had the sharper corners of each super-rhombus as their centre; see Fig. 7.9 with these meandering arches drawn in yellow. The direction in which these elliptical shapes were elongated had a simple if somewhat odd logic: they were elongated either parallel to the edge of the super-rhombus for n = 5, 9, 13,…, perpendicular for n = 7, 11, 15,…, and "circles", or regular polygons, to be quite precise, for n = 4, 6, 8, 10,…

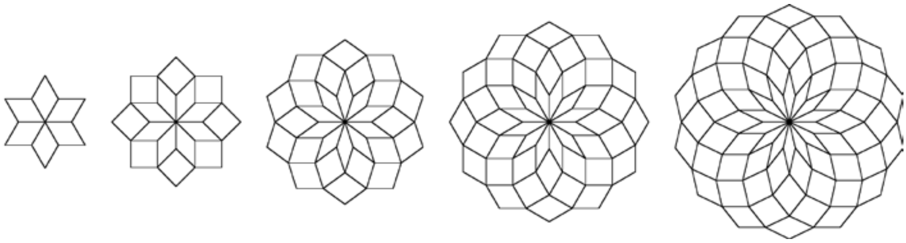


Figure 7.10 The type 2 rose for $n = 3, 4, 5, 6,$ and 7 with their outermost "petals peeled off", producing a concave perimeter. This pattern is $2n$-fold rotationally symmetric.

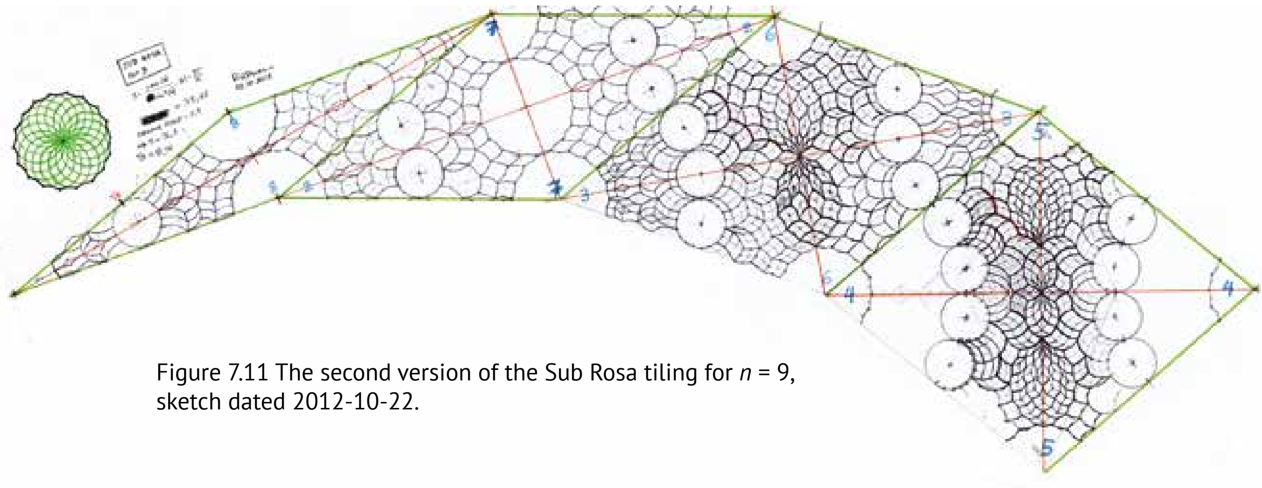


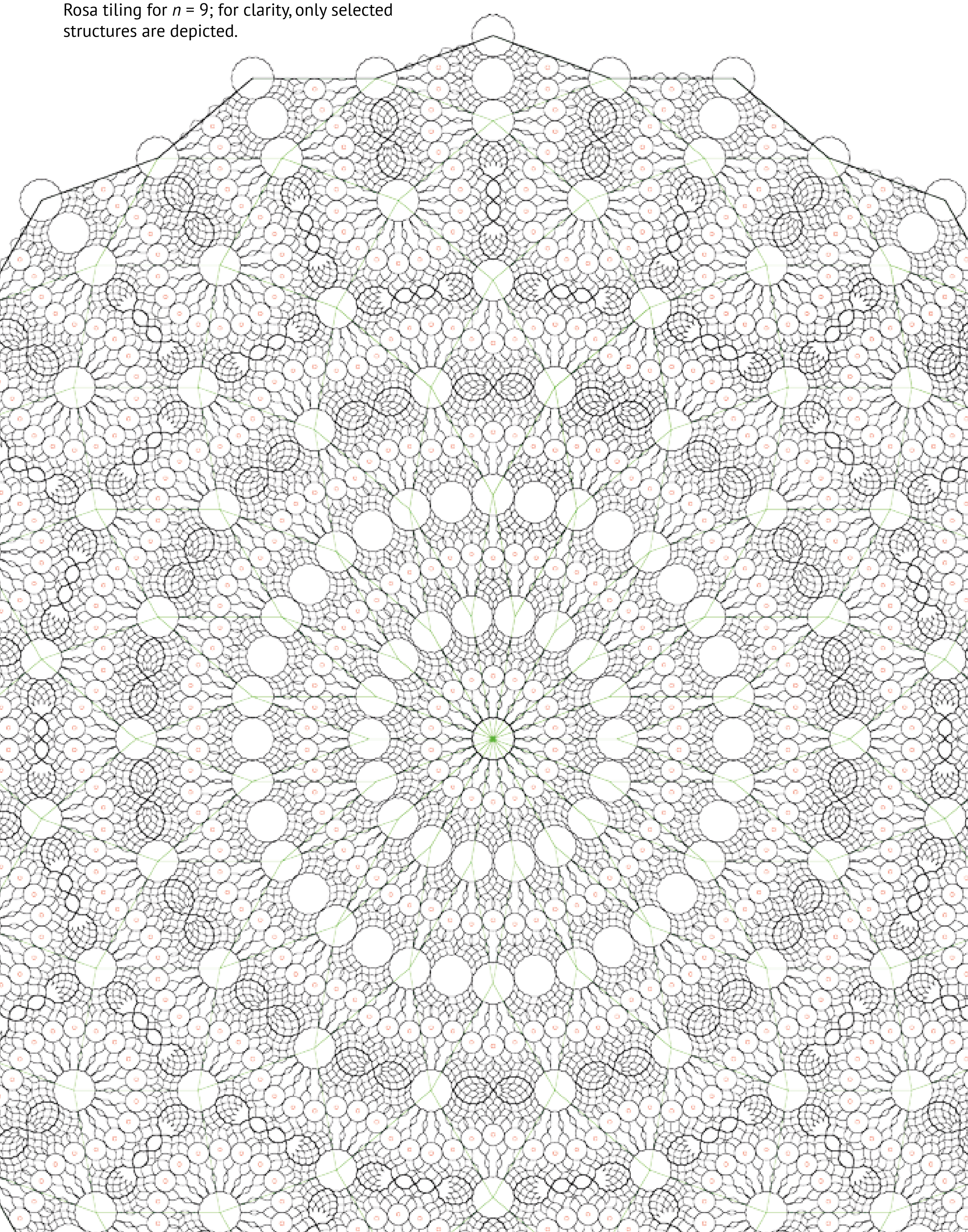
Figure 7.11 The second version of the Sub Rosa tiling for $n = 9$, sketch dated 2012-10-22.

The Second Version

In July 2012 I discovered another version of the Sub Rosa tiling system. In this second version, the length of the edge of the super-rhombus was slightly shortened; that is, the scaling factor S was adjusted, and the outmost rhombuses of the type 2 roses were removed.[395] This second version had several advantages to the first version, not least that the elliptic shapes became circular for all n, and their position slightly moved to fit exactly in between the edges of the super-rhombus; see Fig. 7.11 above.

[395] Please note that with version 1 and 2, I refer to major versions of the whole Sub Rosa system that theoretically included solutions for all n. In other words, versions do not refer to any single Sub Rosa tiling for some n, but to a characteristic set of rules which produces essentially "similar" tilings for all n. Minor developments could even be referred to, for example, as v.1.2.0 etc., but such a detailed notation is unnecessary.

Figure 7.12 The second version of the Sub Rosa tiling for n = 9; for clarity, only selected structures are depicted.



In this second version, the number of circular type 1 roses in between the edges of a super-rhombus actually equals the vertex angle of that particular corner (occupied by a larger type 2 rose): there is one circular type 1 rose near the sharper corners of super-rhombus S(1, *n*–1), two circular type 1 roses near the sharper corners of super-rhombus S(2, *n*–2), three circular type 1 roses near the sharper corners of super-rhombus S(3, *n*–3), etc. This fact gives the second version an especially nice “pedagogic” clarity and also some aesthetic appeal as the arches of these smaller roses meander smoothly through the tiling; see Fig 7.12 with only selected structures depicted.

In Fig. 7.11, the “horn”, made of four green super-rhombuses, is mirrored along the green line at the lower right corner, the sharpest super-rhombus S(1,8) is erased, and the remaining “crescent”, consisting of seven super-rhombuses S(1,8), S(2,7), S(3,6), S(4,5), S(5,4), S(6,3), and S(7,2), is rotated and copied 2*n*-times. This operation produces the complete Sub Rosa tiling with the overall shape of the type 2 rose, which is depicted for the values n = 3, 4, 5, 6, and 7 in Fig 7.10 and for n = 9 in green in Fig. 7.11 (left upper corner).

Such a mirror-copy-paste-rotation operation for the “horn” of four green super-rhombuses seen in Fig. 7.11 is depicted in Fig. 7.12; please notice that Fig. 7.12 is drawn on a relatively large scale; thus, the unit rhombuses cannot be shown, and only selected structures are depicted in such a small image. This image also illustrates how a limited space is a real challenge in depicting a large tiling in a common book or journal sheet: the details are too numerous and small to be properly depicted. A large map-type print gives the best impression of what a “complete” Sub Rosa tiling looks like.[396] One such print is added in the end of this book as Appendix C. In the meantime, I trust that the reader is able to “zoom out” and see the complete circular pattern, for example, in Fig. 7.12.

Let it be emphasized here how the core idea of the Sub Rosa tilings can be seen in Fig. 7.12: the black zigzag-line seen at the upper edge defines the perimeter of the whole tiling, which is in the form of a type 2 rose. The whole pattern can be reduced 33 times and copy-pasted wherever the larger (circular) zigzag-edged roses are left empty. All of the larger white circular areas seen in Fig. 7.12 have the same “internal structure” as the whole pattern; thus, the Sub Rosa tiling, like fractals, is self-similar. The circular areas of the type 2 roses in Fig. 7.12 were left blank for

[396] As “tiling” in the strictest mathematical sense means a complete covering of an infinite plane, no “tiling” can *ever* be depicted. The “complete” refers here to a patch which in visual format contains all of the information needed for how the substitution process proceeds.

the sake of clarity; it is easier, for example, to see the points where the vertices of the super-rhombuses meet. The smaller concatenated circular areas mark where type 1 roses are located.

Organic Forms Appear

The super-rhombuses S(1, *n*–1) and S(2, *n*–2) can be tiled trivially for all *n*, the actual work begins with S(3, *n*–3). In my first version, there were simple "lens shapes" in the centre of each super-rhombus, highlighted in orange in Fig. 7.9, but in the second version, a more intriguing shape emerged. After many experiments, the most natural way of tiling interiors of super-rhombuses turned out to be a shape which resembled a scale of, let us say, a fish, or being more fanciful, a dragon. As in my first version, which used 2x2 "windows" made of four congruent unit rhombuses, the second version also utilized "2x2" shapes, but unlike in the first version, they were not made of four *identical* rhombuses but of two identical and two non-identical rhombuses instead. These fish- or "dragon-scale" shapes are drawn in black in Fig. 7.11, although the four unit rhombuses they each contain are barely visible in the image. There are endless possibilities for arranging these "dragon scales"; that is, mid-structure, to tile the interior of a super-rhombus, that is, superstructure. For some reason, the mid-structures with seemingly "organic" forms always seemed to work the best: they constituted the superstructures in the most logical and elegant manner with the simplest rules of thumb applied.

This emergence of "organic forms" was a most thrilling discovery during this study of substitution tilings. It is still puzzling why such a system that is defined with "hard" geometric rules seems to develop "organic forms" so willingly. Another puzzling issue is the fact that when such organic alternatives are followed, something new and surprising seems to emerge. The drawing in Fig. 7.11 is dated 2012-10-22, which is when a satisfying solution for $n = 9$ was found. The solution came in the form of a "pine cone". It took almost six months for me to realize that this "pine cone" might hold more than meets the eye.

Seven to Eleven

In my rhombic tiling studies, I have always emphasized the odd values of rotational symmetry. Perhaps this has been partly due to Penrose tilings' having the "forbidden" fivefold symmetry and partly due to the "dullness" of the square grid tiling, which for a long time also made the other even values look less appealing to me. Already In the Hex Rosa tilings, the rules are slightly different for even and odd values, and if one tries to figure out their rules, it is more logical and easier to work with either the even or the odd rotational symmetries; at least this was my line of reasoning at the time. In short, from 2012 to circa 2014 I was studying mainly the first odd values; that is, $n = 3, 5, 7, 9, 11$. As I said, just as the Penrose tilings from the 1970s are 5-fold symmetric, my Sub Rosa tiling for $n = 7$ was actually my first real discovery in the field of substitution tilings.

The number of different tiles in rotationally symmetric rhombic tilings grows in steady linear manner as n increases.[397] Also, the number of different super-tiles increases at the same pace. For $n = 3$, the Sub Rosa tiling system includes only one type of tile (1,2); for $n = 5$, the system includes two types of tiles (1,4), (2,3); for $n = 7$, three types of tiles (1,6), (2,5), (3,4), etc. For $n = 11$, the system includes five tiles (1,10), (2,9), (3,8), (4,7), (5,6). For the vertex angles x and y for all rhombuses (x, y), the simple equation $x + y = n$ holds. I will not derive the equation for the Sub Rosa scaling factor here. I will merely state it as a given that for odd n, the Sub Rosa scaling factor $S(n) = \cos(w) / \sin2(w)$, where $(w) = 90°/n$. This gives the following approximate values: S(3) = 3.464, S(5) = 9.960, S(7) = 19.689, S(9) = 32.660, S(11) = 48.972, S(13) = 68.326, S(15) = 91.022, and so forth. Thus, for $n = 11$, for example, the edge of each super-rhombus is 48.97 units long, or 24.49cm if the unit of 0.5cm is chosen as a base.

I noticed that when drawing the tilings by hand, the smallest practical unit for the edge of the unit rhombus was 0.5cm, producing lengths of the edges of super-rhombuses typically in the range of 10…50cm. I chose to always draw these edges in green, and this gives the scale of my drawings seen in Figs. 7.8–7.13. In these drawings, the unit rhombuses drawn with black pen or grey pencil are sometimes barely visible. The tiling becomes a puzzle where not only the type of unit rhombuses increases steadily as n grows, but also the number and area of super-rhombuses, which have to be covered seamlessly with the unit rhombuses. By the beginning of 2013 I had solved all cases up to $n = 11$, but a grand scheme for a general solution was still missing.

Then, in February 2013, it occurred to me that the "pine cones" were not just some arbitrary elements which could be used to tile the diagonals of the super-rhombuses but actually formed a larger, *new type of rose*. Since October 2012 I knew of the possibility of such a patch, but in no super-rhombus was there enough space for such an "oversized" configuration. It was not until the value $n = 11$ and its super-rhombus S(3,8) that there appeared enough room for such a large "type 3" rose to develop in its "natural" environment, that is, as a part of a tiling. With smaller values, there was an unpleasant problem: the type 3 rose did not fit into any super-rhombus if the "pedagogic" arches of type 1 roses were to be preserved.

[397] This holds also for many other rotationally symmetric rhombic tiling systems and not just for the ones I have been developing.

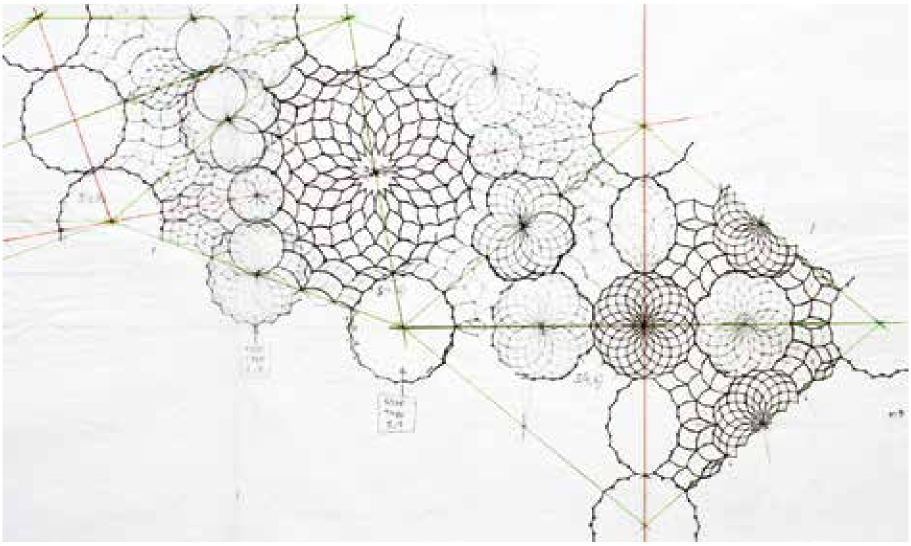
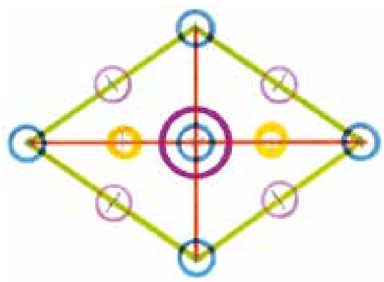


Figure 7.13 Version 3 of the Sub Rosa tiling for n = 9, sketch dated 2014-07-19.

Figure 7.14 A highly simplified schema of a generic Sub Rosa super-rhombus.



Four Roses?

If type 1 rose has radius R, then type 2 rose has radius 2R, and type 3 rose has radius 4R. All types are regular polygons when *complete*, but some rhombuses often have to be removed from their outermost layers. The types of roses and the number of removed rhombuses, or "petals", can be marked, for example, as *t.p*, where *t* is the type of rose and *p* is the number of layers of rhombuses erased from the perimeter. Type 1.0 roses are depicted in Fig. 7.7, and type 2.1 roses are depicted in Fig. 7.10. The large rose seen in Fig. 7.13 (below) is of type 3.3, but its layers are not counted in unit rhombuses but in "dragon scales", which normally contain 2x2 unit rhombuses. Whereas the sides of the "petals" in both smaller roses are straight (one unit rhombus edge), the sides of "petals" of type 3.3 rose are slightly curved inwards (two unit rhombus edges); see Fig. 7.13. The apparent similarity of type 2.1 and 3.3 roses in Fig. 7.13 to, for example, a real chrysanthemum or the centre of a sunflower, has not escaped my attention.

Eventually, the benefits of also having such a type 3 rose in the centre of the super-rhombuses for smaller *n* seemed greater than the benefits of keeping the arch made of type 1.0 roses. The birth of the third version of Sub Rosa tilings can be traced to October 2013, when the type 3.3 rose was introduced in the tiling for $n = 7$, though at the cost of dismantling all of the aforementioned arches made of type 1.0 roses. On the other hand, such rearranging made it possible to have yet another type 2.2 rose in the middle of the edges of super-rhombuses; see Fig. 7.13 for the third version for $n = 9$.

Fig. 7.14 depicts a highly simplified schema for a generic Sub Rosa super-rhombus of version 3. The green edges and orange diagonals bisect all unit rhombuses along them (no rhombuses shown). Blue circles mark type 2.1 roses, pink circles mark type 2.2 roses, and a purple circle that is twice as large in the middle marks a type 3.3 rose. It also seems possible that there are more roses to be found in the area between the pink roses. Yellow circles in Fig. 7.14 mark such a speculative rose. The roses are not so densely located in this highly schematic image as they are, for example, in Fig. 7.13.

I consider it likely that in the centre of a generic super-rhombus $S(x,y)$, as depicted schematically in Fig. 7.14, a type 2 rose can be positioned (blue circle in the middle) for all even $x \geq 2$, and a type 3 rose (the large purple circle) for all odd $x \geq 3$. In each of the four corners of every super-rhombus, there is always a type 2 rose (blue circles). There are also other hypotheses that I am willing to make of Sub Rosa tilings for larger values that I have been unable to investigate by drawing them manually. One hypothesis concerns the possible existence of novel types of even larger roses. As *n* increases, the area of a super-rhombus grows faster than the area of type 1, 2, and 3 roses, producing more and more space in between the roses depicted only schematically in Fig. 7.14. One might be able to construct and locate even larger roses of, let us say, type 4 in a Sub Rosa tiling. Perhaps there even exist an unending series of different types of roses, each approximately twice the size of the previous type.

Another hypothesis concerns the number, type, and order of unit rhombuses along the diagonals of a generic super-rhombus. For the (green) edge, these three parameters are known for all *n*, and for the diagonals, the number and type of unit rhombuses are known, but not their order, which has too many possible permutations to be tested with pen and paper. If we mark in Fig. 7.14 the green edge, or side, S, half of the orange-red horizontal diagonal X, and half of the orange-red vertical diagonal Y, we know for sure that no matter in what order the unit rhombuses are in the diagonals, the lines X, Y, and S always form a right-angled triangle, and thus for their lengths holds the simple Pythagorean equation $X^2 + Y^2 = S^2$.[398]

Contacting the Field

By the spring of 2012 I realized that I had found *something* worth sharing, perhaps even worth publishing, but I didn't know *what* it was exactly. From that time onwards I tried to make contact with mathematical scholars in the field of tiling, and substitution tilings in particular. This turned out to be a difficult task. As a non-professional, I had no experience in the field. One major problem was the correct terminology, or rather my lack of proficiency therein.

In the summer of 2012 in Helsinki, I had the pleasure of meeting Dr. Alan H. Schoen (b. 1924), an American physicist known for his studies of triply periodic minimal surfaces and the discoverer (1970) of one such important object, called Schoen's G surface, also known as a *gyroid*.[399] For decades Dr. Schoen has also been interested in rhombic tiling systems. Our discussions were most encouraging, but they did not lead, for example, to collaboration or a written article. I also had contacts with a few other scholars abroad, who further encouraged my research in tilings. The problem was that I had obtained some results and reached a certain level of understanding of the subject on my own but could not get much further alone.

As luck would have it, at some point I contacted the Finnish mathematician Jarkko Kari, who had published important articles on Wang tiles, mentioned in the previous Chapter. I met professor Kari in October 2012 at the University of

[398] Using this equation, it is possible to numerically check that the drawings are done correctly along the diagonals of the super-rhombuses. Such a procedure is simple but laborious as one needs first to calculate the tables for the lengths of the diagonals of all unit rhombuses and then to check what is the correct combination to satisfy the equation $X^2 + Y^2 = S^2$. Such a puzzle first seemed almost too difficult to be solved, but soon patterns started to emerge, ultimately making this principle an indispensible tool in checking the accuracy of my hand-made drawings for larger *n*. I will not provide the tedious tables for the lengths of the diagonals of the unit rhombuses in this thesis, but the formula for their length is provided in Appendix B, equation (2).

[399] Gyroid surfaces are also found in nature. Extremely porous microstructures of some butterfly wings have the gyroid-form, as do even some structures in the retinal cones of some shrews. See Philip Ball, *Shapes* (2009), pp. 81–90, or "Triply Periodic Minimal Surfaces (TPMS)" on Alan Schoen's own homepage at http://schoengeometry.com where the section "Tilings by Rhombuses" can also be found. Schoen's "Infinite Periodic Minimal Surfaces Without Self-Intersections" NASA Technical Note D-5541, Washington, DC, May 1970, is also available online at http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19700020472.pdf (both accessed 2016-07-29).

Turku and presented the main points of my tiling research to him. Professor Kari immediately understood my systems and estimated that the results seemed correct. Eventually we ended up writing an article together about this substitution tiling and its properties. As the paper is now part of my doctorate thesis (Appendix B), I consider it proper to say something of the roles of its authors.

Roles in the Co-operation

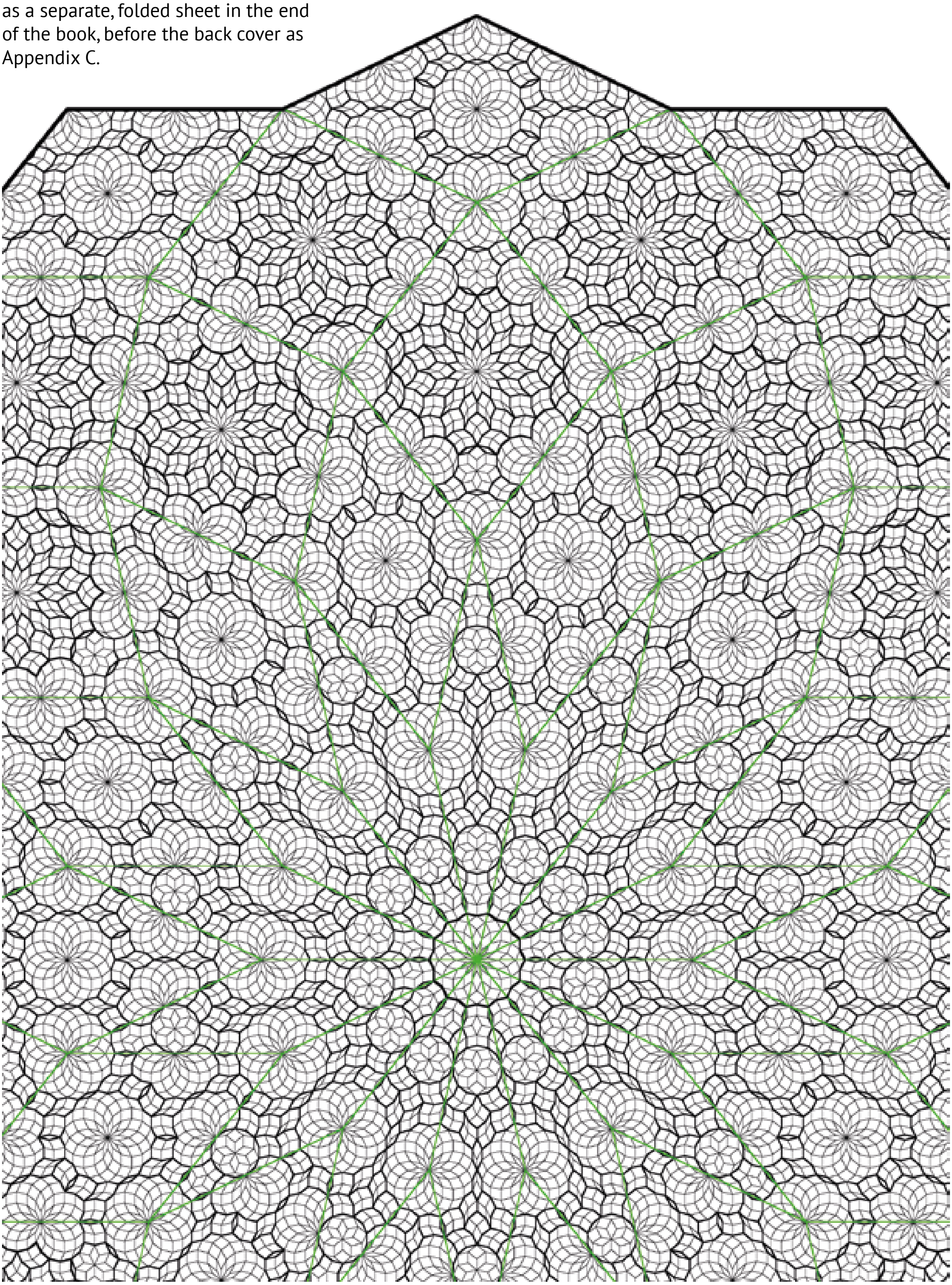
I am the father of the first idea of the system, and I derived the equation for the scaling factor, developed the edge replacement rule, and the idea of using the sum of the diagonal measures of unit rhombuses to verify that the sum equals the length obtained by the edge replacement rule. I also wrote the first version of the article and manually drew the tilings reproduced in the paper, which my wife Henna Helander then drew with ArchiCAD. For quite some time, from late 2013 to late 2014, the article was to be written by Reino Niskanen, a Ph.D. student of mathematics at the University of Turku, and myself, since professor Kari's time constraints prevented him from helping with the writing at that time. In September 2014 Mr. Niskanen moved to the UK to continue his studies there. In the end, it was agreed that the final paper would be submitted with Jarkko Kari and Markus Rissanen as its authors. This seemed correct as the substance of the article started to gain weight after professor Kari took a research sabbatical and had time to work on the paper from spring 2014 onwards. Around the turn of 2014/2015 professor Kari managed to work out a mathematically rigorous proof that the Sub Rosa system works for all n and not just for relatively "small" values like, for example, $n < 100$.[400] I will not go deeper into the proof here; it suffices to say that its main idea is to show that the area inside the green edges can always be correctly tiled for all n with the Sub Rosa replacement rule, see Fig. 7.13 and Fig. 11 in the Appendix B.[401] The idea and construction of this proof is the work of Jarkko Kari, and it takes up the latter half of the paper. As a professional in the field, professor Kari also provided the proper nomenclature and the correct definitions used in the paper. The article was submitted in December 2015, and it was published after a smooth peer-review process first online[402] in April 2016 and then in print in July 2016, volume 55, issue 4 of the *Discrete & Computational Geometry*, pp. 972–996.

[400] Professor Kari wrote a short computer program to test whether my edge replacement rule allowed for a tiling of every super-rhombus interior in such a way that the substitution process could be done correctly for all $n < 100$. Naturally, however, such a test is no proof that there are no problems ahead with some larger values.

[401] The proof developed by Jarkko Kari used results developed by S. Kannan and D. Soroker in 1992, and by R. Kenyon in 1993; see Appendix B and its references [7] and [8].

[402] At http://link.springer.com/article/10.1007/s00454-016-9779-1 (behind a pay-wall).

Figure 7.15 Version 3 of Sub Rosa tilings for n = 7, all unit rhombuses depicted. This image is provided in complete form as a separate, folded sheet in the end of the book, before the back cover as Appendix C.



A Spoiler of Two *Eigenvalues*

I must explicitly state it here that the chances of ever finding any solid matter with its atom structure resembling a Sub Rosa pattern look slim, to say the least. When the paper was about to be published, I informed some researchers in the field of it, and the following is a part of a response I received from Joshua Socolar, Professor of Physics at Duke University, North Carolina, USA: “There are indeed some connections here with problems that have been puzzling me for some time. There is one technical point that is particularly interesting to me: the 7-fold tiling you show has all of the properties that I would expect, including the feature that two eigenvalues of the substitution matrix have magnitude greater than unity. The eigenvalues in your case are {387.669, 4.04491, and 0.285698}. The important implication is that the 7-fold substitution tiling is NOT a quasicrystal; i.e., it does not have strict quasiperiodic translational order.”[403] In short, a real quasicrystallic matter has a quasiperiodic character in an especially strong sense, meaning that it can be seen as a projection from an *n*-dimensional orthogonal hyper-cubic lattice into a two-dimensional subspace. In such a case, it should have only one, and not two, of its substitution matrix eigenvalues larger than one. How these concepts are defined or numerically handled falls outside the scope of this thesis.

The Main Result of the Sub Rosa Tiling System

The existence of an aperiodic set of prototiles was discovered by Robert Berger in 1966. In 1974 Roger Penrose discovered the first aperiodic set of prototiles, which forces a nonperiodic tiling with a fivefold rotational symmetry, and soon afterwards discovered another set that uses only two “decorated” rhombic prototiles. To my knowledge, this was also the first published example of any quasiperiodic tiling with a non-crystallographic rotational symmetry, that is, with a rotational symmetry that is divisible by some other integer than 2, 3, 4, or 6. The Penrose tilings with their fivefold rotational symmetry can be seen to connect very closely to the tilings studied by Dürer and Kepler.[404] In 1982 Dan Shechtman discovered a certain Al-

[403] E-mail from Joshua Socolar (2016-04-08).

[404] There is also a hypothesis proposing that Islamic craftsmen used Penrose-type inflation rules and quasiperiodic patterns to make their tilings. See Peter J. Lu and Paul J. Steinhard, “Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture”, *Science*, Vol. 315 (23 Feb. 2007), pp. 1106–1110. The editor John Bohannon wrote in the same issue that the subject had already generated heated debate. His main point was that Lu and Steinhardt had not mentioned the earlier work by Emil Makovicky, a professor who had studied such tilings for two decades and published some examinations earlier, for example, in Hargittai (ed.) 1992. See John Bohannon, “Quasi-Crystal Conundrum Opens a Tiling Can of Worms”,

Mn alloy possessing a tenfold rotationally symmetric x-ray diffraction pattern and published the discovery in 1984. This was the first example ever of a previously unknown type of solid matter with a non-crystallographic rotationally symmetric atomic structure. Such matter was soon named quasicrystals. To my knowledge, quasiperiodic tilings with non-crystallographic rotational symmetries were found between 1974 and 2016 for values *n* = 5, 7, 8, 9, 10, and 11, but a general solution for all *n* was not known. The Sub Rosa tilings provide such a schema for all *n*, and this is the main result of the mathematical research in this study.[405]

Science, Vol. 315 (23 Feb. 2007), p. 1066.

405 As the second anonymous reviewer of the Sub Rosa article wrote: "This paper proves the existence of planar substitution tilings with 2*n*-fold rotational symmetry for arbitrary *n*. Experts in this area have long believed that such tilings existed, but this is the first rigorous proof that they do." An e-mail (2016-02-23) from Kenneth Clarkson, Editor in Charge of the *Discrete & Computational Geometry*, Vol. 55, to the authors of the paper.

8 | The Rise and Fall of the Basic Forms

In this chapter, I provide a schematic account of how humans have utilized the basic forms to describe nature-related ideas throughout history. The concept of *Anschaulichkeit* is discussed briefly. I also introduce a new concept that generalizes the idea of the basic forms. Additionally, I address the question of whether or not there are fundamental geometric forms or structures which are outside the scope of human comprehension either now or in the future.

As I said earlier in this thesis, the triangle, square, and circle seem to emerge into existence as completely developed figures. There seems to be no trace of evolution left in them from any apparently simpler two-dimensional, regular or irregular, shape or shapes, which could have evolved into these three pure forms.[406] These three shapes are closed and convex two-dimensional flat forms with a simple and well-defined boundary. They seem to contain not a single unnecessary part and no trace of their history. They certainly seem to be relatives of some sort, with each forming nowadays a very characteristic family of their own. From their appearance, it is not evident what their common ancestor looked like, if such an ancestor existed at all. Nevertheless, there must have been some kind of evolution of these forms in some distant past – or did they simply drop from the sky in a complete and finished condition, as the Danish national flag *Dannebrog* is said to have done on June 15th, 1219 in Estonia, during the Battle of Lyndanisse?[407]

406 See Chapter 2, p. 31 in this book, after the subtitle "A Form is Born".

407 Conveniently equipped with a wooden staff enabling King Valdemar II to grasp the flag and lead his army to victory. See, for example the site maintained by the Ministry of Foreign Affairs of Denmark at http://denmark.dk/en/quick-facts/national-flag or see https://en.wikipedia.org/wiki/Battle_of_Lyndanisse for a later depiction of this miracle (both accessed 2016-09-02).

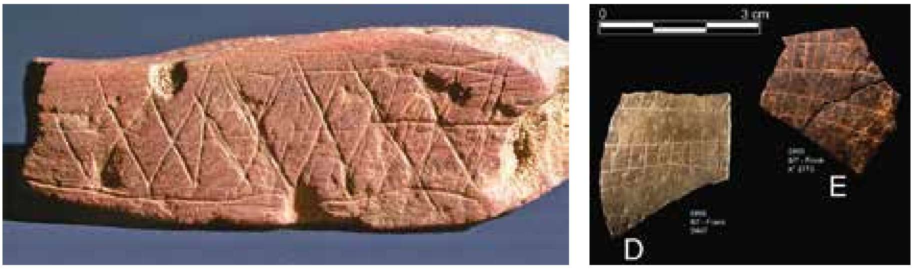


Figure 8.1 A block of reddish ochre, length *c.* 8cm, decorated with geometric lines, from the Blombos cave, marked *c.* 77,000 years ago (left), and *c.* 60,000 years old engraved ostrich eggshells from the Diepkloof Rock Shelter (right), both locations in South Africa.

Order in the House…

But talking seriously, these three geometric forms constitute not the beginning of a long road but the end of it. As far as I can see, this long road, which led to the refinement of these three ultra-pure forms, has got its origin in decorations and ornaments. I thus believe that ornaments and decorations are the cradle of the elementary shapes and geometric patterns cherished by humans. Ornaments provided a culturally stable basis for precise geometric shapes to evolve long before the very concept of geometry was even conceived. In other words, there must have already been a tradition of some kind of "proto-geometric" figures, patterns and simple rules before any more abstract ideas or practical needs became involved. The American metallurgist and historian[408] of science Cyril Stanley Smith (1903–1992) understood this aspect well when he wrote of decorated objects, not of their aesthetic aspects, but of their materials: "It seems that the first and most imaginative use of practically every material was, before quite modern times, in making something decorative. People are experimentally minded when looking for decorative effects, but they can't experiment with established techniques on which their livelihood depends."[409]

Smith was talking about the transition from the Stone Age to the Bronze Age, and his point was that people didn't discover metals, a novel and superior material,

[408] Indisputably, C. S. Smith was at least once himself the focus of making history, not only in academic metallurgy, but in more serious spheres: in the Manhattan project, he was the head of the division which was in charge of the metallurgic properties of all fissile materials used, for example, in the first nuclear bombs Trinity, Fat Man, and Little Boy.

[409] Cyril Stanley Smith, *A Search for Structure; Selected Essays on Science, Art, and History*, 1981, p.110, note 6.

with the purpose of making better tools but rather worked and played with metals in decorations, jewels and other "non-serious" objects. That better tools eventually were produced was just a fortunate coincidence. Furthermore, the latter part of Smiths' idea is very true: normally one avoids taking big risks when one's living and survival are in question. One feels more relaxed while playing with something "less serious" like art and decoration. Might this line of thought also hold in intellectual and conceptual realms? Let us modify Smith's words slightly and say: "It seems that the first and most imaginative use of practically every *form* was, before quite modern times, in making something decorative. People are experimentally minded when looking for decorative effects, but they can't experiment with established *structures* on which their livelihood depends." Intellectual curiosity and free play, much more than the serious struggle for everyday survival, are more likely to produce novel ideas. It may happen that some of those novel ideas *might* develop into something, which in turn *might* also make everyday life easier, or perhaps not.

I argue that it was exactly the realm of ornaments and decorations, which also provided the best platform for the conception of visual order, geometry and symmetry to develop in some distant past. This development became possible exactly because ornament was not such a serious business; it was nothing on which our ancestors' survival depended.[410] When there became the need for some practical geometry to evolve in, for example, building or field surveying, this ornament-based "proto-geometry" was available with a stockpile of useful and already "tested" patterns. The French historian of architecture, Antoine Picon, wrote about the subject in the following way: "The Latin word for ornament, *ornamentum*, shares, for instance, a common etymological origin with the verb *ordino*, meaning to organize, to order, as if an ornament, any well-conceived ornament, expressed the underlying order of things. This mysterious kinship between ornament and ordering is confirmed by another pair of words, 'cosmetics' and 'cosmos'. Both derive from the Greek verb *kosmein*, meaning to adorn as well as to arrange. Again, what seems at stake is the intimate relationship between superficial almost gratuitous-looking appearances covering reality like make-up applied to a face, and the deep structures present under their thin veil."[411]

[410] Tribal marks and religious symbols, for example, might have been a more "serious business".

[411] Antoine Picon, *Ornament; The Politics of Architecture and Subjectivity*, 2013, pp. 37–38. I am grateful to the art historian Anna Ripatti for bringing Picon to my attention. Latin ōrdinā is the first-person singular present indicative of ōrdinäre "to put into order, arrange" coming from ōrdō "row, order, line, rank, series, pattern, routine". The original meaning of the verb ōrdior, "to lay the warp; begin to speak or write; begin", seems to be "to be put in a certain order" the threads which are laid in a pattern to be weaved in the loom. Source: Michiel de Vaan, *Etymological Dictionary of Latin,* 2008, pp. 433–434.

Figure 8.2 With their age of *c.* 17,000 years, these wall paintings in the cave of Lascaux, France, are the oldest known depictions of right-angled grids. The paintings of the cave have badly deteriorated since their discovery (1940), and the colours of these rectangles, for example, are nowadays (2017) barely visible anymore. This image is from the 1955 book *Lascaux, or the Birth of Art* by Georges Bataille.

We can only guess the function of, for example, these multi-coloured grid-patterns, or "blazons", painted in the cave of Lascaux, seen in Fig. 8.2. Often such patterns have been interpreted as magical, tribal or group identity signs, and there are many vertical grid signs, coloured and not coloured, especially in the cave of Lascaux.[412] These patterns and their lines have been partly painted and partly scraped, like so many other works in the cave. There is also variation in the size and exact shape of these rectangles, but the overall impression is of three perpendicular grids or "checkerboards" made of 3x3 rectangular modules; see Fig. 8.2 above.

Since prehistory there has been a slow evolution of conceptual forms and structures developed and used by human cultures. Objects such as the Ishango bone might have played a role in the process. Such objects might have witnessed the first steps in how some forms slowly expanded their existence from the conceptual space of decorative realms into hitherto unknown realms, where information was recorded, preserved and retrieved, not in a spoken but visual format.

As I demonstrated in Chapter 5, all information has a form, or, in other words:

412 Ruspoli (1987), pp. 154–160; 'The Signs and Prehistoric Language'. Aujoulat (2005), p. 66 states that various "non-figurative" images, including dots, lines or marked areas make together 22.1% of the total number of images in the cave of Lascaux.

all *in-formation* has been *in formation*. I am not saying that these very coloured blazons in the cave of Lascaux, for example, would have actually had anything to do with the purpose of storing abstract information. Nevertheless, in them we see some of the seeds of a slow and quiet development of such proto-geometric structures and shapes which later must have developed into conceptual tools which in turn helped to place our worldview into such geometric schemas which we modern humans consider self-evident.

It has been with the aid of such forms and structures that humans have tried to get a cognitive grip on nature and the universe that surrounds us. It is intolerable for the curious human mind not to have some kind of mental image or even vague theory of "how" the world around us is, or *should be*, structured. In building such models for himself, mankind has used myths, symbols and geometric models, among others, in explaining the unexplainable.[413] Often such mythic, religious, symbolic and other models of the world have been more or less fused, and only relatively recently such categories have become separated.

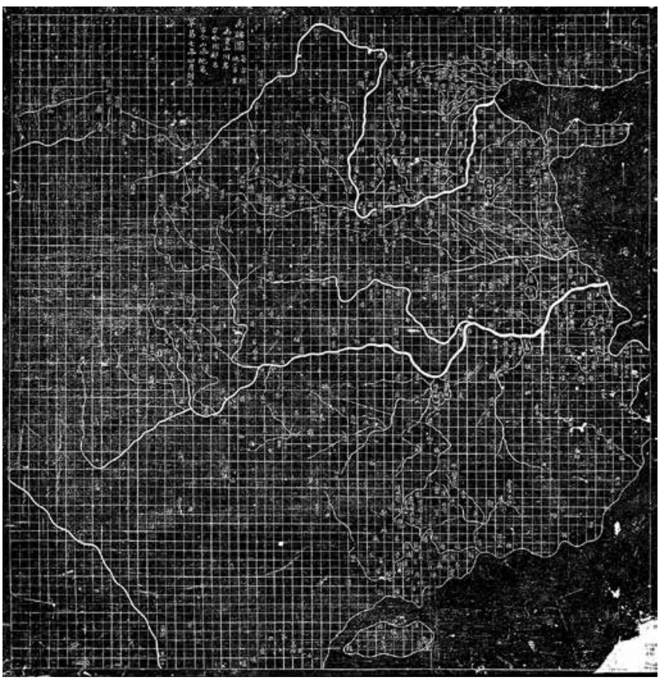


Figure 8.3 The Chinese *Yu ji tu* map, *c.* 84 x 82 cm pencil rubbing on paper of the original stone carved in AD 1136, now located in the Forest of Stone Steles Museum in Xian, China. The whole map is divided into 5110 squares, each corresponding to an area of 100x100 *li*, approximately 50x50 km. On the other side of the same stele is *Hua ji tu,* another map made at the same time and depicting the same area but in a slightly more "oriental" style.

[413] One examination of the relations between geometry and its usages in human culture is Shoji Kato's doctoral thesis *Place of Geometry*, 2015.

… And Harmony in the Universe

One can argue that it was not until Pythagoras (*c*. 580–500 BC) that science in its current form started to develop. Before Pythagoras all philosophy concerning nature was more or less speculative, with no real means of testing which theories were sound and which not. Thales (*c*. 624–*c*. 546 BC), for example, believed that all matter is based on water; Anaximander (*c*. 610–*c*. 546 BC) said the ultimate reality is *apeiron*, Greek for "unlimited, indefinite or infinite"; Anaximenes of Miletus (*c*. 585–*c*. 528 BC) said all is based on air; while Empedocles (*c*. 490–*c*. 430 BC) thought universe was composed of fire, air, water and earth together. One large problem with such "natural philosophy" is that we have no means to tell which theory or opinion is better than some other, or on what grounds.

Pythagoras was the first philosopher who recognized a quantitative law in a natural phenomenon which could be measured and tested. As the story goes, for there are only stories and myths left: one day Pythagoras happened to walk by a blacksmith's forge, from where he heard the rhythmic sounds of hammering.[414] While listening to this hammering, Pythagoras made a strange observation; at certain moments, the sounds of the different hammers hitting at the same time were in perfect consonance and harmony. Such harmony interested Pythagoras greatly, and he entered the smithy to see what was behind this phenomenon. After having studied different explanations for this pleasant consonance, Pythagoras came to the conclusion that it was the weights of the hammers and not, for example, the personal qualities of the blacksmiths which caused the harmonious consonances. Eventually Pythagoras was able to determine that the weights of the hammers were related in the same way that the numbers 6, 8, 9, and 12 are related, and he understood that he had discovered something important. Pythagoras returned home and started to conduct experiments.[415]

"First he attached corresponding weights to the strings and discerned by ear their consonances." Then, "he poured ladles of corresponding weights into glasses, and he struck these glasses – set in order to various weights – with a rod of copper or iron." Finally, "Thus led, he turned to strings, measuring their

[414] Boethius, *Fundamentals of Music*, 1989, pp. 17–19, or see Daniel Heller-Roazen, *The Fifth Hammer; Pythagoras and the Disharmony of the World*, 2011, pp. 11–17. It is interesting what Boethius (*c*. 480–524 AD) chooses as his examples in the very first chapter of his *De institutione musica*, "Fundamentals of Music", a treatise of *music*, when speaking of senses and reason: "Further, when someone sees a triangle or a square, he recognizes easily that which is observed with the eyes. But what is the nature of a triangle or a square? For this you must ask a mathematician." Boethius (1989), pp. 1–2. Perhaps here is an allusion to Boethius himself, as he had written *De arithmetica* and translated mathematical texts by Euclid and Ptolemy, among others, from Greek to Latin.

[415] Boethius (1989), p. 19.

lengths and thickness that he might test further."[416] What Pythagoras found can be demonstrated with one instrument, the monochord, which consists of one string attached over a sound box at both ends, equipped with a bridge that can be moved to divide the string into two parts as wished.[417]

Figure 8.4 A woodcut depicting Jubal, Pythagoras, and Philolaus, from *Theorica musicae,* 1492, by Franchinus Gaffurius.[418] Jubal is a character in Old Testament, described as "the father of all such as handle the harp and organ [flute]." [419] Philolaus (c. 470–c. 385 BC) was a philosopher considered to have been the Pythagoras' successor. In this much reproduced illustration there are not only four but six numbers present: 4, 6, 8, 9, 12, and 16, all multiples of two and three.

[416] *Ibid.*, p. 19.

[417] Heller-Roazen (2011), p. 13. The Finnish architect Aulis Blomstedt, mentioned in Chapter 2, was very interested of arithmetical and musical ratios. He even built a monochord to test ratios, not only with his eyes, but also with his ears, see Sarjakoski (2003), p. 188.

[418] Franchino Gaffurio [Franchinus Gaffurius], *The Theory of Music*, 1992, is an English translation of the Latin original *Theorica musicae* that was printed in 1491–1492 in Milan, the same time as Boethius' *De institutione musica* "Fundamentals of Music" in Venice.

[419] Genesis 4:21; King James Version. Jubal had one sister Naamah and two brothers; Jabal and Tubal-cain, who were described as "the father of all such as dwell in tents, and of such as have cattle", and as "an instructor of every artificer in brass and in iron", respectively. This sounds very much like a pivotal moment in the history of mankind with one group staying in the Stone Age, as nomads and herdsmen, while the other group entering first the Bronze Age and then the Iron Age; eras named after these "artificial" materials. In this division he fate of the fiddler remains unclear.

A vibrating string produces a sound and the length of this string determines the pitch, that is, the frequency of the sound. Experimenting with his monochord, Pythagoras made the following discovery: if the length of a string is divided in two parts in such a way that these lengths are in a simple numerical ratio, the produced tones sound especially pleasing to the ear. [420] Or, with two identical monochords: let the whole string of the first one to vibrate freely and set the string of the second one to such length that these two tones sound satisfying when played simultaneously. In modern terminology these ratios, or musical intervals, are called unison (1:1) [421], octave (2:1), perfect fifth (3:2), perfect fourth (4:3), major third (5:4), and minor third (6:5). Thus, Pythagoras discovered for the first time an exact numeric description for any natural phenomenon, and suddenly there were rules even in invisible nature, laws which could be expressed not only in the language of myths, parables, or philosophical speculations, but in the language of unambiguous numbers.

The Pythagorean school, or fraternity, believed that similar types of ratios of numbers were behind all phenomena in the universe and that nature could be understood and explained by such number relations, or, taken one step further: the Pythagoreans believed that ultimate reality was based on relations of *numbers*, not on water, air, or "apeiron". The Pythagoreans had good grounds for such a belief as their master had shown that such numeric relations truly do exist in nature. Pythagoras is also considered the first to have proved a theorem that nowadays bears his name, even though it was known, for example, to Greek and Babylonian mathematicians centuries earlier.[422]

420 What is told here holds for strings but actually *not* for weights. The attained numeric relations were so simple and beautiful that it went unnoticed for almost two thousand years that the *numbers* change when the *situation* changes. Surprisingly, it was Vincenzo Galilei, the father of Galileo Galilei, a lutenist and musical theorist by passion and a wool merchant by necessity, who in 1589 published his treatise in which he explained that the harmonic proportions derived from strings must be squared for the numbers to hold for masses. In other words: for masses, the octave corresponds not 2:1 but 4:1, the fifth not 3:2 but 9:4, and the fourth not 4:3 but 16:9; see Heller-Roazen (2010), p.67, or Claude Palisca, "Scientific Empiricism in Musical Thought", in Hedley Howell Rhys (ed.), *Seventeenth Century Science and the Arts*, 1961, pp. 127–129.

421 The ancient Greeks said that two tones with a similar pitch were in "unison", tones one octave apart in "diapason", tones one fifth apart in "diapente", and tones one fourth apart in "diatesseron"; see Heller-Roazen (2010), p. 14. In the modern diatonic scale, the interval of one octave is divided into seven parts, or *notes*. From any note to a corresponding note one octave higher or lower, eight steps are needed; hence *octave*, which comes from the Greek word ὀκτώ [oktō] meaning "eight". A *second* is an interval spanning two consecutive notes in the modern diatonic scale, a *third* an interval spanning three consecutive notes, etc., up to the interval spanning *eight* consecutive notes, which constitute the interval of one octave.

422 The Pythagorean theorem says that in every right-angled triangle, the sum of the squares of two shorter sides equals the square of the longest side, that is, the hypotenuse, or, expressed as an arithmetic equation: $a^2 + b^2 = c^2$.

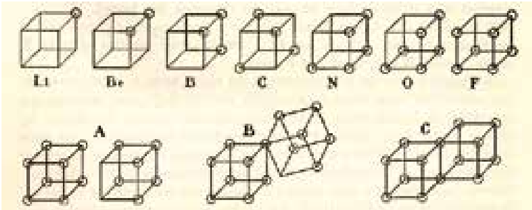
Figure 8.5 A medieval model of the universe depicted in Petrus Apianus' *Cosmographia* (1524). This image is from the 1564 printing.

A World of Simple Structures

This worldview based on numeric relations was not forgotten even during the Dark Ages. In the Middle Ages and during the Renaissance, in addition to the theory of music, or actually in perfect unison with it, the field of astronomy[423] cherished the idea of a universe with harmonious numeric relations. What was sub-lunar, under the orbit of the Moon, was the Earth made of four elements; water, earth, air and fire, a world in disorder, always in change and under corruption, but what was super-lunar, beyond the orbit of the Moon, was a world made of finer substance, a world in perfect order, including the seven planets with their eternal, harmonic movements. Beyond the planets was firmament, the eighth sphere of the fixed stars, after which came the ninth heaven of pure crystal, beyond which came the never-ending, fiery *Empyreum*, the habitat of the eternal God and his angels, archangels and the blessed souls, plus or minus a few other bombastically-named spheres on the outskirts of this imaginary terminus. Naturally, such a universe was easily depicted with concentric circles; see Fig. 8.5 above.

Johannes Kepler showed that the orbits of the planets were not actually perfect circles but ellipses. The ellipse, like the circle, is also a relatively simple geometric

[423] Considerable parts of Kepler's *Harmonices Mundi* (1619), for example, consists of musical notes positioned on the lines of staff, presenting celestial music produced by the planets.



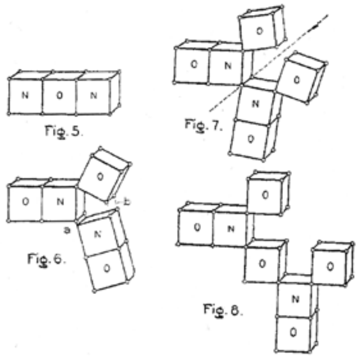


Figure 8.6 Cubic atom models by Gilbert Newton Lewis from 1902 but published only in 1916 (above), and models of connected cubic atoms by Irving Langmuir from 1919 (right).

form that our mind is able to handle, not only with geometric, but also numeric means. It is amazing how recently simple geometric shapes have been used to model some of the most fundamental structures of nature. One such case is the models of atoms proposed at some point in chemistry, where their actual composition was not yet known and not even considered as being as important as their connectivity with other atoms and molecules. One interesting class of such geometric models is "cubic atoms" proposed by, for example, Gilbert Newton Lewis (1875–1946) and Irving Langmuir (1881–1957) in the beginning of the 20th century; see Fig. 8.6 above.[424] As we now know, such cubic models of the atoms were soon replaced by more advanced models, developed by Niels Bohr (1885–1962) and Arnold Sommerfeld (1868–1951), among others.[425]

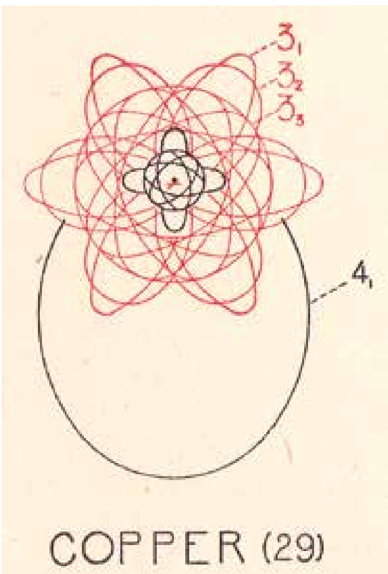


Figure 8.7 The Bohr-Sommerfeld model of the copper atom, from around 1920 (left). In this type of model, all electrons were assumed to revolve around the nucleus in elliptic orbitals that were exactly defined by fixed parameters.

[424] Lewis and Langmuir managed to develop not only cubic atom models and the theory of chemical bonds, but also the difficult relationship in connection to these issues and their priorities; see, for example, Patrick Coffey, *Cathedrals of Science: The Personalities and Rivalries That Made Modern Chemistry*, 2008, pp. 134–150. Also, Kragh (2012), pp. 118-122.

[425] Helge Kragh, *Niels Bohr and the Quantum Atom*, 2012, 'The Bohr-Sommerfeld Atom', p. 176.

There were two particularly large conceptual leaps involved in the development of physics in the beginning of the 20th century. These conceptual jumps came with the general theory of relativity by Albert Einstein (1879–1955), and quantum mechanics, developed by several physicists. Both provide a cornerstone to modern physics and the way the fundamental characteristics of nature are nowadays theoretically understood and technologically utilized. Some aspects of classic physics had already provided challenges for visualizations, but these challenges were nothing compared to what was coming.

In Einstein's general theory of relativity, for example, physical reality is no longer described in the Newtonian frame of three spatial dimensions plus absolute and universal time[426], which is ticking in every part of the universe with a uniform pace, totally ignorant of whether there is something in that space or not. In Einstein's theory, time and space can no longer be separated as they form a four-dimensional space-time unity. Time is no longer independent of the material world – quite the opposite: mass bends four-dimensional space-time, slowing "local" time and bending three-dimensional space, both phenomena that have been verified in numerous experiments since the first one successfully conducted by Arthur S. Eddington (1882–1944) during the solar eclipse of 29th May 1919.[427] Even if it might be difficult for us to imagine such a four-dimensional amalgam of space-time, its components – space, time, and dimensions *per se* – are comprehensible, and we can form some kind of mental image of their amalgam, even if such an image wouldn't be correct in all aspects.

In quantum physics, however, things even stranger have been witnessed. In quantum physics, the very concept of causality comes under question, and certainties are replaced by probabilities. In addition, there are fundamental limitations to precision concerning the properties of the observed particles,

[426] There are interesting analogies between such models with or without absolute time and medieval cosmology. In the medieval universe, there is also time in the sub-lunar world, but it seems to be of a different sort than time in the super-lunar realms. Mankind is able to act in this sub-lunar time, whereas he is only an observer, at best, of the more solemn time of the super-lunar spheres. Also, the outermost heaven with the Almighty is not only a place beyond *space* but also a place beyond *time*, as it is the place of the everlasting eternity. In the Newtonian model, this solemn super-lunar absolute time covers the whole universe, Earth included, with no interaction with matter. In the "Einsteinian" worldview, on the other hand, it is as if sub-lunar, "material" time expands to cover the whole universe and is inherently chained with matter. If there is eternity in such a world, it is not located on the outskirts of the universe, but in a point-like "singularity" that is assumed to exist at the centre of every black hole.

[427] See, for example, Arthur S. Eddington, *Space, Time and Gravitation: An Outline of the General Theory of Relativity*, 1920, Chapter VII: 'Weighing Light', pp. 110–122.

limitations that were introduced in 1927 in the *uncertainty principle* by Werner Heisenberg (1901–1976).[428] Also, the very ideas of matter and radiation seem profoundly (con)fusing since the discovery of the wave-particle duality, culminated in the work of Louis de Broglie (1892–1987). Other phenomena as well, which are not covered by classic physics, have been experimentally verified, such as "quantum jumps" and "quantum entanglement". In short, with the discovery of quantum physics, some fundamental questions about the character and comprehensibility of nature have become salient.

Anschaulichkeit

Arthur I. Miller, Professor of History and Philosophy of Science, has written of such comprehensibility in relation to modern physics in his book *Imagery in Scientific Thought: Creating 20th-Century Physics*, 1986, especially in Chapter 4 'Redefining Visualizability.'[429] Another philosopher of science, Henk W. de Regt, has written of the subject in his articles "Erwin Schrödinger, *Anschaulichkeit*, and Quantum Theory" in 1997 and "Spacetime Visualisation and the Intelligibility of Physical Theories" in 2001.[430] Both authors write of a concept which is manifested in the German words *anschaulich*, *Anschaulichkeit*, and *Anschauung*. Suzanne Gieser writes in her 2005 book *The Innermost Kernel: Depth Psychology and Quantum Physics. Wolfgang Pauli's Dialogue with C. G. Jung*: "The translation of the term *Anschaulichkeit, anschaulich*, and *Anschauung* is not altogether easy. *Anschaulich* is used together with the German word for picture (*Bild*), and refers to the classical visual models, i.e. the atoms as a planetary system. […] *Anschaulichkeit* is linked to the possibility of grasping an abstract reality beyond our perceptions."[431] Even if, as de Regt said: "the notion of *Anschaulichkeit* played a crucial role in the genesis of quantum mechanics"[432], the concept is by no means a creation of modern physics.

In her introduction to *Goethe's Botany*, Agnes Arber wrote: "We know that Goethe's actual visual impressions were peculiarly intense, and greatly influenced his mode of thought; indeed, his inclination always drew him to 'picture thinking'.

428 See, for example, Werner Heisenberg, *The Physical Principles of the Quantum Theory*, 1930, Chapter II, §1: 'The Uncertainty Relations', pp. 13–46.

429 First edition by Birkhäuser (Boston) in 1984, reprinted by MIT Press in 1986, Chapter 4: pp. 127–177.

430 Both papers were published in *Studies in History and Philosophy of Modern Physics*, Vol. 28, No. 4 (1997), pp. 461–481, and Vol. 32, No. 2 (2001), pp. 243-265, respectively.

431 Suzanne Gieser, *The Innermost Kernel: Depth Psychology and Quantum Physics. Wolfgang Pauli's Dialogue with C. G. Jung*, 2005, p. 68, footnote 228. In Finnish, the matter is simpler: *anschaulich* is "havainnollinen", *Anschaulichkeit* is "havainnollisuus", and *Anschauung* can be translated as "näkemys, käsitys, (mailman) katsomus".

432 De Regt (1997), p. 462.

For this way of apprehending nature, [Troll, 1926] uses the expression 'intuitive Anschauung', which might be called, 'thinking with the mind's eye'; it lies midway between sensuous perceptions reached through bodily sight, and the abstract conceptions of the intellect."[433]

Some developers of early quantum physics, Erwin Schrödinger perhaps most of all, were worried about losing the *Anschaulichkeit* in proposed models, while some others, such as Werner Heisenberg, were not so concerned about the issue.[434] Then there were others, such as Wolfgang Pauli (1900–1958), who saw the demand for *Anschaulichkeit* almost as an obstacle to the development of physics. In his letter to Bohr on 12th December 1924, Pauli wrote: "I consider this certain – despite our good friend Kramers and his colourful picture books – 'and the children, they love to listen.' Even though the demand of these children for *Anschaulichkeit* is partly a legitimate and a healthy one, still this demand should never count in physics as an argument for the retention of fixed conceptual systems. Once the new conceptual systems are settled, then also these will be *anschaulich*."[435]

Conceptual Systems

In the following section, I introduce another kind of conceptual system which, I argue, can be used in classifying geometric shapes and structures not only in practices related to culture but perhaps also within our mental categories. This classification is not a general scheme for all conceptual systems met in every – potentially or actually – existing geometry. Even less is it a classification for "conceptual systems" completely unrelated to visual forms. All of the classes and forms that

[433] Arber, "Introduction to Goethe's Botany", *Chronica Botanica*, Vol. 10, No. 2 (Summer 1946), p. 85. Also, the Swiss pedagogue and educator Johan Heinrich Pestalozzi (1746–1827) was interested in *Anschauung*, and he wrote about the subject in his book *ABC der Anschauung*, published in two parts in 1803. Pestalozzi's approach was very stiff, formal, and *very* non-visual; the essence of his book was to teach children to divide a line into n and a square into n x n equal parts. His follower Friedrich Froebel (1782–1852), on the other hand, was superbly innovative in using geometry in his pedagogy. Froebel's fascination with simple forms and geometry had a solid basis; he studied and worked with crystallography seriously enough to be offered a professorship of mineralogy from Stockholm in 1816, which he somewhat surprisingly declined in favour of his work in pedagogy. See, for example, Norman Brosterman, *Inventing Kindergarten*, 1997, pp. 20–25.

[434] De Regt (1997), *passim*.

[435] De Regt (2001), p. 252, where the original reference is given as Wolfgang Pauli, *Wissenschaftliche Briefwechsel, Band I: 1919–1929*, New York: Springer-Verlag, 1979. The 'colourful picture book' to which Pauli refers is *Bohr's Atomtheori*, 1922, by H. A. Kramers and Helge Holst, from which the colourful Fig. 9.7 in this Chapter is scanned, or, to be more precise, from its 1923 English edition *The Atom and the Bohr Theory of its Structure*. Such a schema of elliptic orbitals around the nucleus has showed its imaginative power in fields of visual culture, where it has become *the* symbol of the atom.

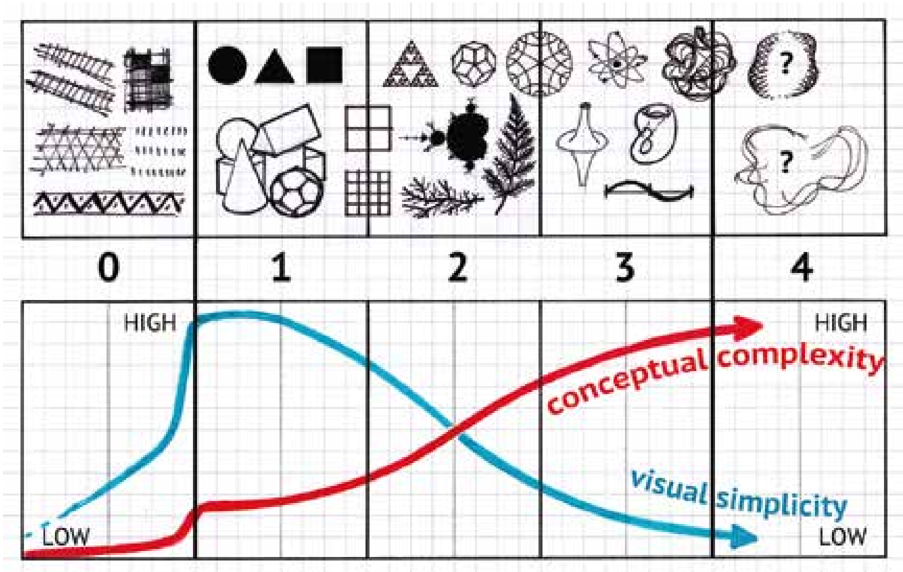


Figure 8.8 One plus four classes of Conceptual Systems (CS) in geometry: CS0, CS1, CS2, CS3, and CS4.

the following conceptual systems contain are, or can be seen as, nature-related.

For simplicity's sake, in the following I shorten "conceptual system in geometry" to CS. In addition, I enumerate these classes and speak of CS0, CS1, CS2, CS3, and of a hypothetical CS4. In Fig. 8.8 above, it would have been possible to depict these classes as a hierarchical pyramid or as enlarging circles, with each new one including the previous ones, or as a tree-shape with the "root class" evolving and dividing into separate branches. The possible visual shape(s) of this categorization is not relevant as such, however; what matters is the idea that such a classification can be made in a meaningful way at all. Also, the boundaries of these categories are not always totally sharp, as there are shapes and structures which can be taken to belong to two categories that form a bridge from one class to another. As there is always some kind of shift from one paradigm to another involved in the evolution of ideas, it is possible to see the borders between the classes as representing the liminal areas for such shifts. It would be possible to call these classes of conceptual systems as "paradigm families" for different types of geometries.

One could also speak of conceptual "platforms" which can be seen as a natural habitat for a specific family of forms, structures and patterns. In some instances, a "space" can also be considered as a "form" if seen from higher dimension, the surface of the sphere, for example, is a "space" for the two-dimensional observers on it, while seen from the three-dimensional "outside", it is a "surface." Nevertheless, I try to avoid speaking of "conceptual spaces"[436] and keep my presentation to forms, perhaps even to "basic forms", which constitute the kernel of each such conceptual class.

CS0: Geometric Forms Before Geometry

This class is located in prehistory, when certain, either crude or more sophisticated, geometric shapes already existed, but before the *concept* of geometry itself came into existence. As mentioned before, geometry got its name from the art of measuring fields. I believe some laws of geometry had already been observed much earlier in repeating geometric ornaments. Only after this incubation period was the idea of geometric *facts* transformed into other realms of culture, and used, for example, in measuring fields.

CS1: Geometry as a Conceptual System

This class was born when geometry was first seen as a *conceptual* system in its own right. In other words, it was no longer subservient to the aesthetic aims of decoration or tribal or magical activities. [437] I believe that most probably at some point, the clarity and precision of the patterns and shapes within the CS0 reached such a level that it enabled and triggered the development of the concept of geometry in the human mind. Somebody figured out not only the intelligible rules in geometric patterns, but also realized that such rules are independent of any *specific* pattern or its material manifestations. After such a conceptual jump, visual simplicity, precision and clarity also increased, at least potentially, and, seen from the human perspective, I argue, reached their practical maximum; see Fig. 8.8. I also argue that CS1 is the most fundamental class of geometric forms for humans. This class is the one we most probably think of when we think of "geometry" and its objects. This class certainly includes simple geometric forms such as the circle, triangle and square. From these simple forms, we can expand this first real class of geometry by way of variations, combinations and generalizations.

In addition to rectangles, trapezoids and rhombuses, completely new forms were soon constructed to be included in this class, forms such as ellipses and

[436] In Peter Gärdenfors' book *Conceptual Spaces; The Geometry of Thought* (2004), for example, "conceptual spaces" are not nature-related in any way. With "conceptual spaces" Gärdenfors refers to well-defined co-ordinate systems into which some cognitive models are positioned according to their quality dimensions. These qualities may be time, colour, sound, taste, form, etc. The dimensions of such representations vary greatly, starting from the classical representation of time as a one-dimensional line with past, present, and future.

[437] Even if it retained its utilitarian name, which referred to land-measuring activities and not to abstract realms, into which geometry had just entered.

all regular polygons as a logical generalization of the equilateral triangle, square, pentagon, hexagon, heptagon, etc. In addition to plane figures, all simple solids, such as the cube, sphere, cone and pyramid, must be considered to belong to this class as well as some two-dimensional shapes that are not spatially limited, such as the hyperbola and parabola. All forms in this class typically have a rather simple visual appearance, and all of these forms can be defined with a few simple – or relatively simple – constructional descriptions. To sum up: in CS1, a small amount of data produces simple visual results.[438]

As previously mentioned, all of these classes and the forms they contain are, or can be seen as, "nature-related". In the case of CS1, this includes, for example, atomic order in crystals and different diagrams representing the functions of nature, of which I gave examples in Chapter 4. Also included are, as demonstrated in Chapter 2, the forms in category CS1 and their elements: points, lines, planes and volumes, which were much used in the visual arts for teaching drawing and painting. They were used to deconstruct the visible forms of nature into simpler ones, a method which Cézanne explicitly mentioned in his famous letter.

But just by looking at nature, it becomes obvious that such a system of basic shapes is far from perfect, much less a comprehensive system for describing nature and its visible forms. This shortcoming was clearly expressed by Lancelot Law Whyte, who described it in 1955 in the following way:

> *For if a God created this universe in His own image, He certainly was not a Pythagorean. Once we recognize this fact, the garden loses its magic and begins to look like a museum. As we glance around we see how narrow the selection has been: there are five Platonic solids; the circular or elliptic paths of the planets; few geometrical models; some vibrating strings; patterns made out of the integers: 1, 2, 3, 4, …; and not very much more. There is no butterfly or fish or scorpion; no nude; nothing organic, subject to reproduction, growth, and decay.*[439]

Whyte did not see the emergence of this "science of tomorrow", or even parts of it, which he so longed for. What he anticipated in 1955 sounds very much like a theory containing irregular, "organic" forms, or, in other words: fractals.

438 After the conceptual system of such geometry was developed it reached a new level of abstraction and theoretical rigidity in the axiomatic method introduced by Euclid around 300 BC. Nevertheless, as my emphasis is mainly in the visual aspects of geometry and basic forms, I will not go further into the subject of "axiomatic geometry".

439 Lancelot Law Whyte, *Accent on Form: An Anticipation of the Science of Tomorrow*, London: Routledge & Kegan Paul, 1955, p. 21.

CS2: Fractals and Self-Similar Forms

This class of forms includes objects such as recursive systems, self-similar patterns[440], cellular automata, L-trees, chaotic and non-linear systems, and perhaps its best-known members, fractals, already introduced in Chapter 4. In fractals and other repeating self-similar forms, there is a difference in the approach of their construction compared to that of the shapes in Euclidean geometry. In classical geometry, the visual complexity of a certain form is typically related to the complexity of the information needed to define the shape. With fractals, cellular automata, and self-similar patterns, this is not the case. In many such cases, the forms are defined with a very few rules or a small amount of initial data, but the results can be visually extremely complex. Think, for example, of the Mandelbrot set; it can be defined with just a few lines, and yet its visual richness and complexity is never-ending and simply stupendous, or, as the mathematician John H. Hubbard said, it is "the most complex object in mathematics."[441] Most probably the *visual simplicity* has reached its minimum limit (so far) in the Mandelbrot set.[442]

Naum Gabo (1890–1977), a notable Russian-born constructivist sculptor of the 20th century, wrote of geometric forms used by the Constructive School:

> [I]*t may seem that we are using forms, shapes, and lines taken from science; but those who think so forget that all forms, shapes, all lines, elementary as well as complicated, geometrical as well as so-called "free", are neither the privilege nor the invention of science. The square and the circle and the triangle and all the rest of them were present in the human observation and served as an expression in works of art of the earliest time. As a matter of fact, it can be asserted that it was actually from the artist that the scientific mind borrowed them and then started to use them as a means of investigation and calculation.*[443]

[440] Also some self-similar structures exist in the CS1 category of classical geometry. Think of, for example, the square grid system, or even a one-dimensional line divided into repeating units, where each element can be divided into a few similar but smaller elements. Furthermore, what else is our decimal system, for example, than such a self-similar structure, where each unit is divided into ten similar but smaller units, which in turn can be divided further into ten smaller units, etc.? The infinite straight line is also "self-similar" as such, as it can be zoomed in or out as much as desired, and yet it always looks the "same". Thus, self-similarity alone is no guarantee of producing "exotic" forms.

[441] A. K. Dewney, *Scientific American*, August 1985, p. 20.

[442] Please note: in Fig. 8.8, the position of the Mandelbrot set should not be taken too literally; in some respects, classes CS2 and CS3 could even change places in this *linear* schema if we want to read it in a more chronological order.

[443] Naum Gabo, *Art and Science*, in Kepes (1956), pp. 62–63.

I agree with Gabo that basic geometric forms such as the square, circle, and triangle most probably were first used in art and especially in ornaments, but I don't agree with Gabo when he says that *all* forms, shapes, and lines, whether elementary or complicated, geometric or 'free', are neither the privilege nor the invention of science. What Gabo writes about reflects the transition from CS0 to CS1 and operates with the geometric forms of CS1, but his observation does not withstand critical examination in the cases of CS2 or CS3, where the forms are mostly, if not completely, the invention of science, not art. To sum up: in class CS2, a small amount of data produces complex visual results.

CS3: Advanced Models of the Physical World

For over two millennia the boundaries of CS1 were seen as those of geometry itself. Even such an eminent thinker as Immanuel Kant thought that the space we live in, not only is, but also has to be Euclidean, by logical necessity alone.[444] Nevertheless, by the beginning of the 19th century, alternative, non-Euclidean geometries started to emerge, first as an unpublished study by Carl Friedrich Gauss (1777–1855) in Germany, and finally in printed form in 1829 and 1832 by the Russian Nikolai Lobachevsky (1792–1856) and the Hungarian János Bolyai (1802–1860), respectively.[445] Such non-Euclidean geometries include what are often called *hyperbolic* and *elliptic* geometries, marked in Fig. 8.8 with a round "hyperbolic disc" (bordering categories CS2 and CS3) and a "pseudo-sphere" (below the hyperbolic disc and the "atom"), which has a surface with a constant ***negative*** curvature (the surface of the actual sphere having a constant ***positive*** curvature).

The German mathematician Bernhard Riemann (1826–1866), among others, further developed the theory of non-Euclidean geometry at the University of Göttingen.[446] Riemann's work in geometry proved to be central in the later development of the geometric model used in Einstein's theory of relativity as it turned out that our three-dimensional space is not "flat", as in Euclidean geometry, but rather "bent" if seen from the perspective of higher dimensions. Other strange geometric forms and objects were also constructed in 19th century mathematics,

[444] Nevanlinna (1968), pp. 20–21. Also the philosopher Michael Friedman writes how it was essential for the Kant's theory of a priori knowledge of the empirical world that three types of space; mathematical, perceptual, and physical space were not distinguished from each other as they are in the contemporary conception of space, geometry, and experience. Friedman calls this modern position "explicitly anti-Kantian". See Michael Friedman, "Kant on Geometry and Experience", in Vincenzo De Risi (ed.), *Mathematizing Space; The Objects of Geometry from Antiquity to the Early Modern Age*, 2015, pp. 275–307.

[445] Boyer (1968), pp. 585-588.

[446] ***Ibid***., p. 588.

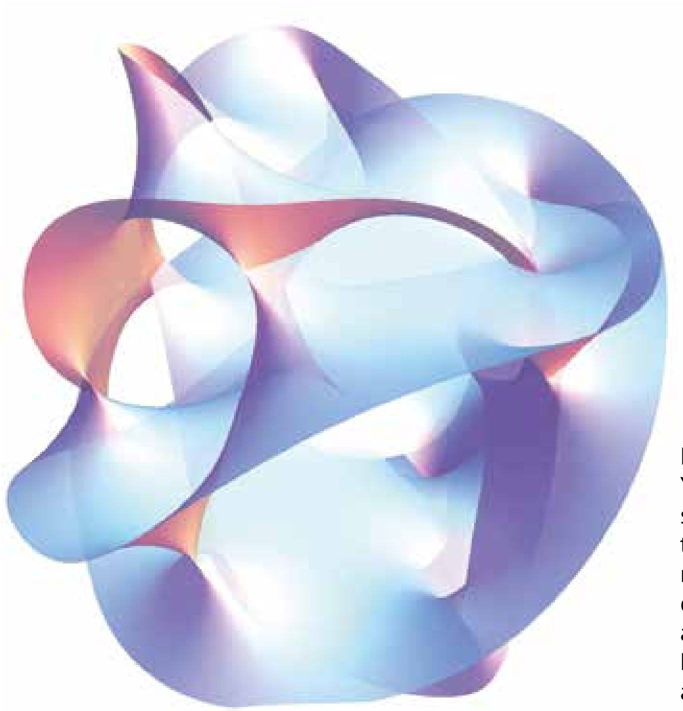


Figure 8.9 A Calabi–Yau manifold, or a simplified and opened two-dimensional representation of the object that exists in an eleven-dimensional hyperspace wrapped around itself.

one example being the “Klein bottle”, which is an object in four-dimensional space made by connecting two Möbius strip together along their edges with the property of having a smooth and completely closed surface but without a separate inside and outside.[447] In three-dimensional space, such an object cannot be constructed without its surface crossing itself at least once. The Klein bottle is also depicted in Fig. 8.8 (below and right of the “atom”).

Even if these “strange” forms and geometries don’t sound too relevant from the point of view of our everyday life, on many occasions they have turned out to represent aspects of our physical world at its most fundamental levels. Thus, many such forms and models are deeply nature-related or, to be more precise, many forms in this system have been specifically developed as advanced models describing the physical world. One such case is the model of the atom we have already looked at. Of course, “advanced” is a relative term. Advanced models of atoms today refer to

[447] Thomas F. Banchoff, *Beyond the Third Dimension*,1990, pp. 178–198.

models of the structure of their nuclei or even smaller parts. The models of atoms by and large are nevertheless excellent examples because in their history, we can see a gradual shift from visually and intuitively comprehensible, that is, *anschaulich* models to more advanced models with much less or barely any visualizability left. In this class, there are models that have a definite geometric shape, whereas from some matrices or equations of quantum mechanics, it is very hard, if not altogether impossible to make visually meaningful representations. And even if some models in CS3 are still geometric by their description, their forms can be *very* complicated.

A good example of such a complicated form is the "Calabi-Yau manifold", which plays an important role in superstring theories.[448] These vigorously vibrating manifolds are "wrapped around themselves" in an 11-dimensional hyperspace.[449] These objects, or spaces, can be mathematically described in very precise terms, but not correctly visualized in our three-dimensional space – not to mention a flat image – as one can easily understand.

These wrinkled entities might just be the form of the very fabric out of which our three-dimensional physical space is made. We are not able to perceive these hypothetical objects as they are "wrapped up" like a flat piece of paper that is squashed into an unspecified-shaped-ball-like object and observed from miles away; what is seen is a "point" at best, just as at best, we "see" elementary particles as points at the moment.[450] This third class of forms differs from the previous two in that typically, all forms or structures in CS3 are developed to describe physical nature. Forms in the two other classes can also depict forms of the physical world, and often they actually do, but often such depictions were not their initial aim. To sum up: in this class, complicated theories produce results that are very difficult to present correctly in a visual form.

[448] Brian Greene, *Kätketyt ulottuvuudet*, a Finnish translation of *The Elegant Universe, Superstrings, Hidden Dimensions, and the Quest for the Ultimate Theory*, 1999; *passim*.

[449] *Ibid.* I will not even try to claim that I would understand what all this actually means, but I do understand that such models have been very seriously studied in theoretical physics for several decades.

[450] This may change in the future. It is possible that the energies available in the Large Hadron Collider at CERN could be high enough to reveal such hidden structures from elementary particles, all of which still behave in the experiments as if they were tiny "points".

CS4: Where Are the Limits?

We are not able to see galaxies with our unaided eyes; we need a powerful and high quality telescope to observe them. Nonetheless, we tend to regard telescopes and microscopes easily as mere technical aids to augment our natural capabilities of seeing. For us, there is nothing mysterious or suspect in such instruments. The way they work and what they do is more or less clear for every educated adult in any developed country. When we turn our attention to even more powerful and complicated instruments such as scanning transmission electron microscopes (STEM), telescopes used in x-ray astronomy, the Laser Interferometer Gravitational-Wave Observatory (LIGO), and gigantic machines such as the Large Hadron Collider (LHC)[451] at CERN, we face more difficult questions about seeing and observing.

It is no simple thing to explain how the "images" are obtained from these instruments. The further we go into technological development, the thinner the connection between human perception and the produced image becomes. The image becomes more "artificial", and it requires more and more sophisticated theories and technical "tricks" to obtain an "image". Thus, the image becomes more a composition of several observations interpreted within some theoretical framework than the result of any single observation or perception. The Canadian-born American linguist, psychologist and semanticist S. I. Hayakawa (1906–1992) expressed this development in 1956 in the following manner:

> [E]*vents at nuclear, atomic, and molecular levels, cosmic ray phenomena, and events at the level of the extremely large, as in astrophysics, are not visual experiences, but logical and mathematical derivations from instrument-observations and hypotheses. These inferred structures and events, then, are never directly experienced; they can only be visual*-ized *through the construction of models* […][452]

It seems as if our culture has reached the point where it is capable of producing theories and models of the structures and forms of the physical world at its most fundamental levels but has no technical means of verifying those models

[451] If the Mandelbrot set were called the most complex object in mathematics, so has the LHC at CERN sometimes been called the most complex machine ever built.

[452] S. I. Hayakawa, "Domesticating the Invisible", in Kepes (ed.), *The New Landscape in Art and Science*, 1956, p. 65. Note; "*visual*-ized" in the original: visual-*ized*.

by experiments or observations.[453] On the other hand, the complexity of those theories and the forms they postulate seem also to have reached such heights where their visualizability and intelligibility are on the verge of being humanly impossible.[454]

All previous conceptual systems of geometry have enlarged our understanding, not only of geometry and its forms as such, but also of our understanding of nature. The major changes in the paradigms concerning nature seem to be somehow deeply connected with the major changes in the paradigms concerning geometry, but which is the *cause* and which is the *effect*? What if the major changes in the paradigms concerning geometry and its "basic forms" come first, and these changes not only make it possible to perceive, but also actually cause the major changes in the paradigms concerning nature?

Nature-related basic forms may have started to develop in places such as Blombos, Diepkloof, Ishango or Lascaux, and they may end with superstring theories if the very concept of "form" loses all meaningful interpretations in such an infinitesimal scales of physics. However, if the geometric descriptions of nature move on from superstring models to the next level of complexity and conceptuality, there will be – once again – a change in the paradigm of *how* we see the bonds between geometry and nature. This means that in the future, there will simply emerge yet another class – CS4 – an even more advanced and more abstract conceptual system of geometry that describes nature, whatever the "basic forms" in such a class might be.

453 One lucidly written survey of the subject is John D. Barrow's book *Impossibility; The Limits of Science and the Science of Limits*, Oxford University Press, 1998.

454 In recent years, in the field of abstract number theory, there has been an interesting case of the Japanese mathematician Shinichi Mochizuki (b. 1969). Mochizuki has made several remarkable contributions to his field in recent years. In 2012 he made available the radical theory that, if correct, would be a sensational revolution in the field. The problem is, however, that his theory uses such novel and radical techniques that even the experts in the field have serious problems in understanding his paper, to say the least. Mochizuki himself said of this difficulty: "it is quite possible to achieve a reasonably rigorous understanding of the theory within a period of a little less than half a year", but for about the last five years experts in the field have held meetings and symposiums to study Mochizuki's paper without reaching a consensus of its correctness. It is possible that a similar situation may one day arise also in physics concerning the geometric nature of the universe. See, for example, the article at http://www.nature.com/news/the-biggest-mystery-in-mathematics-shinichi-mochizuki-and-the-impenetrable-proof-1.18509 (accessed 2017-04-11).

Conclusions

Compendious Snail! thou seem'st to me
Large Euclid's strict epitome;
And in each diagram dost fling
Thee from the point unto the ring.
A figure now triangulare,
An oval now, and now a square,
And then a serpentine, dost crawl,
Now a straight line, now crook'd, now all.
Richard Lovelace (1618–1657), *The Snail*.[455]

I started this thesis by introducing two modes of depicting nature: In the first mode, we use perceptual forms; in the second, conceptual. By the notion "perceptual forms", I am referring to such visible forms and structures as we can see in nature, either with our bare eyes or with the aid of instruments, that is, forms which we can mimetically represent. By the notion "conceptual forms", I am referring to forms, often geometric, which are invented, produced or constructed by humans to visualize phenomena or principles of nature in data-oriented, functional, systematic or "scientific" ways, not by using mimesis but rather abstract concepts. Essentially, by depicting perceptual forms, we aim to show how nature *appears*, and by depicting conceptual forms, we aim to show how nature *works*; the former is made for the eye, the latter for the mind.

One major concept behind this thesis was basic forms. Such a concept is familiar and self-evident for visual artists but not necessarily for people in other fields. I have anchored my study in the circle, triangle and square. In addition, I have

[455] See https://www.poetryfoundation.org/poems-and-poets/poems/detail/44655 for the complete poem (accessed 2016-09-26). This is the second verse of the poem, spelled in the original as "The Snayl". Lovelace's poem came to my attention via Marjorie Hope Nicolson's work *The Breaking of the Circle; Studies in the Effect of the 'New Science' upon Seventeenth-Century Poetry* (1960), p. 61.

argued for the tree-shape to be included in this set of basic forms. Even if such visually simple forms no longer play a leading role in the cutting-edge scientific or artistic study of nature – neither on the perceptual nor conceptual level – there are still good reasons to study their characters and histories.

Let me return one final time to the famous quote by Galileo about the basic forms and nature: "Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics and its characters are triangles, circles, and other geometrical figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth." Basic forms, such as the circle, square, and triangle have provided a tool with which we have been able to comprehend nature, on perceptual and conceptual levels. Such forms have provided descriptions, explanations and visualizations of many natural phenomena.

Because such forms are so effective, expressive and simple, they have also been used extensively in different fields of human culture: in notations, symbols and magic, to name a few examples in addition to the visual arts and science. Because of their certain physical and logical properties, some basic forms are also found in material nature, and not only in its artefactual representations.

In this thesis, I have proposed that the idea and set(s) of basic shapes have developed from the aesthetic and intellectual interactions between human minds and repeating or "regular" patterns which were often very probably decorative in function. Later the human mind began to study, not only geometry and its forms as such, but also on what kind of philosophical and logical assumptions such systems rest. Classic Euclidean geometry, for example, seems so natural and logically unavoidable to us that it is small wonder that any alternative geometries ever developed.

As explained in the previous Chapter, Immanuel Kant, for example, considered the nature of physical space we live in to be Euclidean, by logical necessity alone. Kant also had a firm philosophical position on the order and regularity perceived in nature: "The order and regularity in the appearances, which we entitle *nature*, we ourselves introduce. We could never find them in appearances, had not we ourselves, or the nature of our mind, originally set them there."[456] I see no point in studying form, order or regularity on the premise of a categorical division between

[456] As quoted in Agnes Arber's *The Mind and the Eye,* 1954, p. 85. Original in *Kritik der reinen Vernunft* (1781), p. 134.

Figure Z: Alexander Cozens, *A New Method of Assisting the Invention in Drawing Original Compositions of Landscape* (1785), Plate 14.

nature and the human mind. Evolution has shaped the human mind to cope with the challenges posed to our species in the harsh conditions of the past. We might as well say: "The order and regularity in the appearances, which we entitle *nature*, she herself introduces. We could never find them in appearances, had not *nature* herself, via our mind, set them there."

The ability to see meaningful patterns where there are none has been utilized in the visual arts for ages.[457] The Russian-born British landscape painter, Alexander Cozens (1717–1786), for example, developed a complete method based on using accidental blots as tools for creating landscapes.[458] Cozens acknowledged that

[457] Such as the subsequent use of Rorschach ink blot figures.

[458] Alexander Cozens, *A New Method of Assisting the Invention in Drawing Original Compositions of Landscape*, London, 1785. Selected parts of the book are available online, for example, at https://www.fulltable.com/vts/aoi/c/blot/a.htm (accessed 2016-10-06). Cozens' work is also mentioned, for example, in Nina Samuel (ed.), *The Islands of Benoit Mandelbrot: Fractals, Chaos, and the Materiality of Thinking*, 2012, p. 30. See also Charles A. Cramer, "Alexander Cozens's New Method: The Blot and General Nature", *The Art Bulletin*, Vol. 79, No. 1 (March 1997), pp. 112–129.

already Leonardo da Vinci had written of using such arbitrary blots to excite the imagination.[459]

It is true that the human mind has a strong tendency to see form, order, regularity and meanings where there perhaps is none. Even if we humans have a strong tendency to see – and imagine – objects relevant or threatening[460] to us, such as faces, animals, beasts or monsters in the dark shadows or in arbitrary blots, it is not often that one sees the forms of nature in such pure Euclidean terms as the English writer Richard Lovelace (1618–1657) did in his poem "The Snail", the second verse of which is cited in the beginning of this Chapter.

The basic forms I have been discussing in this thesis have parallel existences; they exist in our imagination, concepts and culture as well as in physical nature independent of us. It seems a happy coincidence that these two realms sometimes meet. The observation and contemplation of physical nature and its forms feeds the imagination and creativity of the human mind, which in turn, every now and then, is able to formulate geometric models and structures which turn out to also exist in nature without our prior knowledge or perception. One example of such a form are the cone shapes: ellipses, hyperbolas, and parabolas. These forms were already studied in ancient Greece, but it was only two millennia later, in the 17th century, through the works of Kepler and Newton, among others, that they were found to exist in nature as forms of the orbits of the planets and the comets.

Another example of such a form is the Penrose tilings, with their quasiperiodic and non-crystallographic five-fold rotational symmetry. Ten years after their initial construction, in the field of mathematics, similar structures were found to exist in certain metal alloys, defying the well-established principles of classic crystallography. Very soon such exotic solid materials, which did not comply with

[459] "[W]hen you look at a wall spotted with stains, or with a mixture of stones, if you have to devise some scene, you may discover a resemblance to various landscapes, beautified with mountains, rivers, rocks, trees, plains, wide valleys and hills in varied arrangements; or again you may see battles and figures in action; or strange faces and costumes, and an endless variety of objects, which you could reduce to complete and well-drawn forms. And these appear on such wall confusedly, like the sound of bells in whose jingle you may find any name or word you choose to imagine." Leonardo in *Libro della Pittura* [The Practise of Painting], *The Notebooks of Leonardo da Vinci*, a reprint after 1970, p. 254, paragraph 508.

[460] John D. Barrow, for example, gave a simple explanation for the human tendency to pay attention especially to symmetric forms: "If we look at the natural environment we see that the lateral (left–right) symmetry is a very effective discriminator between living and non-living things in a crowded scene. You can tell when a living creature is looking at you. This sensitivity has a clear survival value. It enables you to recognize potential predators, mates, and meals." Barrow (1998), p. 5.

the classic rules of crystallography, were named “quasicrystals”. In the first case of the conic sections, we can say that the forms in question reside somewhere between the perceptual and the conceptual realms; we are not able to perceive orbitals as complete forms at any particular moment in time, but on the other hand, such shapes are not purely conceptual forms even if their detection and observation requires abstract theory and astronomic calculations. In the second case, one can say more clearly that quasicrystalline structures are perceptual forms seen in nature, mostly with the aid of some instrument, but in rare cases, even with the naked eye.

I believe that no scientist would be willing to say that there are rules, principles or qualities which would *guarantee* that some hypothesis concerning natural phenomena must hold for sure. We have no way of giving the final verdict on a theory based only on its internal characteristics; it is tests and experiments which give the final proof of whether a theory or hypothesis is true or false.

The same holds with forms created by the human mind in comparison with forms found in nature. We have no *a priori* means of telling whether a form, structure, or pattern invented by humans will someday be discovered in nature as well. For a long time simple geometric forms provided a good means of describing nature in a precise and quantitative way. It was relatively late when the algebraic equations replaced geometric models as more practical and effective tools in the natural sciences. Even if simple geometrical shapes may no longer provide the best possible way of describing nature, they are nonetheless elementary to the human understanding of space and the forms positioned within it.

Basic forms are not the end of a road – quite the opposite. They form a kind of basic level from which all other types of system must be learned or constructed. In the 17th century it was still possible to see nature as written in the language of triangles, circles and other two- or three-dimensional geometric figures, but nowadays, it seems to be written in the language of the Calabi-Yau manifolds, wrapped around themselves in a mysterious and non-perceivable eleven-dimensional space.

No final or complete “theory of forms” is possible. There are several independent and partly overlapping fields where it is meaningful to use the concept of basic forms. Even in these different realms, the specific basic forms are not necessarily the same as they are in other fields. From the human perspective, it is the simple geometrical forms such as the circle, square, and triangle which are most naturally called “basic forms”. These forms have often been treated and explicitly named as *the* basic forms, especially in 20th-century Bauhausian modernism. As 20th-century science has developed, it has been less and less common for any simple form to have a role in significant scientific ideas or discoveries. One notable exception to this is the Penrose tilings. The reason I have given tilings a relatively large space in this thesis is the result of my own studies in the field and those obtained during my doctoral studies. My study shows that even in the 21st century, it is still possible for an artist, using relatively simple basic forms, to obtain results which even have scientific interest.

In writing about such a wide and complex subject, it has not been possible to stay within the limits of any single discipline. Experts in various disciplines can surely point out mistakes and omissions which no scholar in those particular fields would have made. Furthermore, the nature of the subject is such that the major part always seems to remain outside any finished study. As a rule, the Conclusions section is no place to present any further arguments. Nevertheless, I have one more to make: I argue that this work has been well worth doing.

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Introduction
Fig. A http://dl.ub.uni-freiburg.de/diglit/caramuel1670ga (XLb Tafel 16a), p. 0089 in the Universitätsbibliothek Freiburg PDF pagination
Fig. B MR own work
Fig. C http://www.deutschefotothek.de/documents/obj/88966489 orig. p. 28

Chapter 1
Fig. 1.1 MR own work
Fig. 1.2L https://web.archive.org/web/20160511114252/http://www.rainbowint.co.uk/?xhtml=xhtml/services/leakdetectiontraceaccess.html&xsl=services.xsl this is An Internet Archive Wayback Machine page as the original is no longer functional
Fig. 1.2R MR own work
Fig. 1.3 MR own work
Fig. 1.4 MR own work
Fig. 1.5 Garland (2003), p. 41
Fig. 1.6 MR own work
Fig. 1.7 Kepler (1619), a detail of the Plate IV, Liber II, between orig. pp. 58–59, https://archive.org/details/ioanniskepplerih00kepl online and PDF p. 81

Chapter 2
Fig. 2.1 David Bergamini, *LIFE Science Library: Mathematics,* Time-Life International Nederland, 1963, p. 38, original photo: Sol Mednick
Fig. 2.2 http://www.vam.ac.uk/content/articles/l/lunettes/
Fig. 2.3 https://archive.org/details/practicalperspec00th (Plate 1)
Fig. 2.4 MR own work
Fig. 2.5L http://digital.slub-dresden.de/en/workview/dlf/17139/1/0/
Fig. 2.5R http://digi.ub.uni-heidelberg.de/diglit/lairesse1701bd1 orig. p.3
Fig. 2.6L&R http://digi.ub.uni-heidelberg.de/diglit/lairesse1701bd1 orig. pp. 7 and 12
Fig. 2.7 https://archive.org/details/analysisofbeauty00hoga (the title page)
Fig. 2.8L&R https://archive.org/details/naturalprinciple00hayd orig. pp. 14 and 17
Fig. 2.9L&R https://archive.org/details/mthodedecomposit01gras orig. pp. 6 and 7
Fig. 2.10L&R Droste (2014), pp. 109 and 111
Fig. 2.11L Droste (2014), p. 35
Fig. 2.11R https://www.amazon.co.uk/Bauhaus-Jeannine-Fiedler/dp/3848002760
Fig. 2.12L https://www.amazon.com/Bruno-Munari-Square-Circle-Triangle/dp/1616894121
Fig. 2.12R http://web.archive.org/web/20150305204009/http://www.ace-cae.eu/ Internet Archive Wayback Machine page; the A.C.E. logo was changed in June 2015
Fig. 2.13 photo: MR
Fig. 2.14 photo: MR
Fig. 2.15 Barnes (2007), Color Plate 39, Folio 18v.

Fig. 2.16L Fludd (1617), Vol. 1, orig. p. 329,
https://archive.org/details/utriusquecosmima01flud orig. p. 329
the same pagination is used also in the PDF-version
Fig. 2.16M Weigel (1705) http://digi.ub.uni-heidelberg.de/diglit/weigel1705 Plate "ba"
Fig. 2.16R Hogarth (1753), Plate II, Fig. 116,
http://www.metmuseum.org/art/collection/search/404839
Fig. 2.17L&R http://digi.ub.uni-heidelberg.de/diglit/schoen1920
pp. 0025 and 0040 in the Universitätsbibliothek Heidelberg PDF pagination
Fig. 2.18 http://digi.ub.uni-heidelberg.de/diglit/schoen1920
p. 0005 in the Universitätsbibliothek Heidelberg PDF pagination
Fig. 2.19 http://www.pharaon-magazine.fr/actualites/agenda/lart-du-contour-lexposition-v-nement-au-mus-e-du-louvre photo: Sandra Steiss
Fig. 2.20L https://commons.wikimedia.org/wiki/File:Doryphoros_MAN_Napoli_Inv6011.jpg photo: Marie-Lan Nguyen (2011)
Fig. 2.20M https://archive.org/details/hierinnsindbegri00dure online p. 9, PDF p. 15
Fig. 2.20R http://ecx.images-amazon.com/images/I/61L1n6-FZ%2BL.jpg
Fig. 2.21 Sarjakoski (2003), p. 147
Fig. 2.22L https://classconnection.s3.amazonaws.com/736/flashcards/1021736/jpg/vitruvian_man1329152756565.jpg
Fig. 2.22R https://archive.org/details/hierinnsindbegri00dure online p. 115, PDF p.128

Chapter 3
Fig. 3.1 http://lhldigital.lindahall.org/cdm/ref/collection/earththeory/id/10242 Plate 7
Fig. 3.2L http://vwhl.soic.indiana.edu/villa/database(static).php photo: Musei Vaticani
Fig. 3.2R MR own work
Fig. 3.3 Kepler (1619), a detail of the Plate IV, Liber II, between orig. pp. 58–59,
https://archive.org/details/ioanniskepplerih00kepl online and PDF p. 81
Fig. 3.4 Kepler (1596),
Fig. 3.5 Kepler (1611),
Fig. 3.6L Bentley (1931), p. 182
Fig. 3.6R Nakaya (1954), Fig. 18, p. 16
Fig. 3.7 https://archive.org/details/newsystemofchemi01daltuoft
Plate 3, orig. between pp. 218–219, online and PDF p. 235
Fig. 3.8 MR own work

Chapter 4
Fig. 4.1L Haeckel (1862), plate 24, fig. 9.
Fig. 4.1R http://www.radiolaria.org/species.htm?division=64&sp_id=86
Fig. 4.2 Prochnow (1934),
Fig. 4.3 Strüwe (1955), p. 26
Fig. 4.4 Round, Crawford, Mann (2000), pp. 184, and 218, and http://ina.tmsoc.org/Nannotax3/index.php?dir=Coccolithophores
Fig. 4.5L Round, Crawford, Mann (2000), p. 639
Fig. 4.5R http://pubs.rsc.org/en/content/articlelanding/2014/ta/c4ta03750b#!divAbstract
Fig. 4.6 https://commons.wikimedia.org/wiki/File:2780M-pyrite1.jpg Image: Carles Millan
Fig. 4.6 (3) Mottana, Crespi, Liborio, *Minéraux et Roches,* (1977), 24, 31,
Fig. 4.7L http://media.wiley.com/Lux/40/167940.image5.jpg
Fig. 4.7R https://commons.wikimedia.org/wiki/File:Anatomical_Directions_and_Axes.JPG
Fig. 4.8 http://ilmicroscopio.blogspot.fi/2014_03_01_archive.html
Fig. 4.9 http://www.bbc.co.uk/guides/zsjfyrd#z2cx6sg (video)
Fig. 4.10 http://www.redorbit.com/news/space/1345799/earths_magnetic_field_does_strange_things_to_the_moon/

Fig. 4.11 http://sci.esa.int/science-e-media/img/c8/Rosetta_OSIRIS_NAC_comet_67P_20140803_1.png
Fig. 4.12L https://blogs.csiro.au/ecos/a-game-of-hives-native-bees-at-war/ Image: Tim Heard
Fig. 4.12R Verma (1953), p. 98
Fig. 4.13 Marten (1977), p. 147
Fig. 4.14L MR own work after Aujoulat (2005) p. 130
Fig. 4.14R https://en.wikipedia.org/wiki/File:Tree_of_life_by_Haeckel.jpg
Fig. 4.15 MR own work
Fig. 4.16L Parpola (1993), p. 170
Fig. 4.16R Bredekamp (2005), p. 15
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Fig. 4.18 http://www.marekfiser.com/Projects/AnimatedPythagoras
Fig. 4.19 Manfred Schroeder, *Fractals, Chaos, Power Laws* (1991), Dover, p. 9
Fig. 4.20 https://commons.wikimedia.org/wiki/File:WeierstrassFunction.svg
Fig. 4.21 MR own work
Fig. 4.22 https://en.wikipedia.org/wiki/Mandelbrot_set
Fig. 4.23 Klee (1961), Vol. 2, p. 66

Chapter 5
Fig. 5.1 Birkhoff (1933), p. 33
Fig. 5.2 Garner (1970), p. 155
Fig. 5.3 MR own work
Fig. 5.4 MR own work
Fig. 5.5L Klee (1961), Vol. 1, p. 61
Fig. 5.5R Blum (1967), Fig. 6, p. 369
Fig. 5.6 http://fp.optics.arizona.edu/nofziger/OPTI%20200/Pictures/SEM%20CD.png
Fig. 5.7L Sarjakoski (2003), p. 184
Fig. 5.7R http://www.gutenberg.org/ebooks/20239
Fig. 5.8 https://afrolegends.com/2013/08/29/the-ishango-bone-craddle-of-mathematics/
Fig. 5.9L http://www.e-codices.unifr.ch/en/list/one/bbb/0347 p. 24v
Fig. 5.9R Helsingin Sanomat 2008-09-19
Fig. 5.10L E. J. Nyström, *Graafinen esitys ja nomografia* (1947), Otava, p. 57
Fig. 5.10R *A-luettelo*, Th. Wulff Oy, (1940?), p. B33
Fig. 5.11 https://commons.wikimedia.org/wiki/File:32-column_periodic_table-a.png
Fig. 5.12 http://www.academia.dk/Blog/wp-content/uploads/PeriodicTable_LIFE1949.png
Fig. 5.13 http://www.glencoe.com/sec/science/chemistry/mc/pow/images/Image46.gif
Fig. 5.14 Mazurs (1974), pp. 79 and 126
Fig. 5.15 Mazurs (1974), pp. 49 and 87

Chapter 6
Fig. 6.1 Dürer (1525), p. 181
Fig. 6.2 Dürer (1525), p. 146
Fig. 6.3 Kepler (1619), between pp. 58–59; see the footnote 329
Fig. 6.4L Kepler (1619), between pp. 58–59
Fig. 6.4TR Dürer (1525), p. 67
Fig. 6.4BR MR own work
Fig. 6.5 MR own work
Fig. 6.6 MR after Grünbaum & Shephard (1987), p. 21
Fig. 6.7L Kepler (1619), between pp. 58–59
Fig. 6.7R Penrose (1979), Fig. 7, p. 34
Fig. 6.8 Grünbaum & Shephard (1987), p.539
Fig. 6.9 Grünbaum & Shephard (1987), p.543

Fig. 6.10 Grünbaum & Shephard (1987), p.543
Fig. 6.11 Grünbaum & Shephard (1987), p. 544
Fig. 6.12 MR own work
Fig. 6.13 MR own work
Fig. 6.14 Shechtman *et al*. (1984),
Fig. 6.15 Smith & Burke (1967)
Fig. 6.16 Abe (2012), Fig. 21, p. 6797
Fig. 6.16A Kepler (1619), orig. p. 61
Fig. 6.16B Ishimasa *et al*., (2004), Fig. 3a, p. 4

Chapter 7
all images by MR

Chapter 8
Fig. 8.1L https://www.nsf.gov/od/lpa/news/02/pr0202images.htm
Fig. 8.1R http://www.pnas.org/content/107/14/6180.full.pdf
Fig. 8.2 Bataille (1955), p. 110
Fig. 8.3 https://www.loc.gov/resource/g7821c.ct001493
Fig. 8.4 Palisca (1985) p. 228
Fig. 8.5 https://archive.org/details/cosmographiaapia00apia p. 13
Fig. 8.6L http://scarc.library.oregonstate.edu/coll/pauling/bond/papers/corr216.3-lewispub-19160400.html
Fig. 8.6R Kragh (2012), p. 233
Fig. 8.7 Kramers (1923), Plate II
Fig. 8.8 MR own work
Fig. 8.9 https://commons.wikimedia.org/wiki/File:Calabi_yau.jpg

Conclusions
Fig. Z https://www.fulltable.com/vts/aoi/c/blot/a.htm

Markus Rissanen

Summary of the doctoral thesis in fine arts

Basic Forms and Nature

From Visual Simplicity to Conceptual Complexity

This thesis is an interdisciplinary study of basic forms in art and science. It combines artistic and scientific modes of research and consists of artistic productions and the cultural-historical study of forms as well as mathematical explanations.

In the beginning of the research process, I focused more on the theme of scientific visualization. I was interested in how we give visible form to observable phenomena, which nevertheless cannot be depicted mimetically. The paintings that make up the artistic part of the thesis clearly testify to this phase of my research. Some of these paintings, which have been exhibited and pre-examined as parts of my doctoral thesis, are reproduced in the written part of the thesis, which focuses on questions concerning basic forms and nature. The basic forms that I begin with are the simple forms of classic Euclidean geometry, i.e. the circle, square and triangle. In addition to these simple forms, the branching tree-like form, or dendrite, is considered a basic form in the thesis. This last form also provides a bridge to fractals, a class of non-Euclidean forms often found in nature. Rhombuses play a significant role in the last parts of the thesis and in the appendices and are thus treated as basic forms in this study. The circle, square and triangle provide a natural starting point for the study since they make up the very set of forms most often found in the history of teaching visual arts, especially the elements of drawing. This triad constitutes the kernel of modernistic Bauhausian design, while being the embodiment of the Platonic concept of eternal and indestructible ideas as well. Even if these precise geometric forms are prevalent only in the artefacts of human culture, they also are met in huge numbers in physical nature independent of us.

I introduce two new concepts: *perceptual forms representing nature* and *conceptual forms representing nature*. The former refer to mimetic depictions of visible forms and structures that we can see in nature, either with our bare eyes or by using some kind of instruments. The latter refer to seemingly artificial and often geometric forms that we humans have invented by accident or constructed with purpose to visualize phenomena or functions of nature in systematic, theoretical, data-oriented, or, broadly speaking, "scientific" ways. By depicting perceptual forms, we aim to show how nature *appears*, whereas by constructing conceptual forms, we aim to show how nature *works*. In other words, the former are mostly made for the *eye*, and the latter for the *mind*. This thesis also reconsiders the concept of basic forms and introduces the alternative generalized notion of *conceptual systems of geometric forms*. Such systems utilize simple geometric schemes and are built on previous, simpler systems. I show how a chain of conceptual systems reaches from prehistoric "proto-geometry" to the super-string theories of modern physics.

In addition to the paintings and the cultural-historical study of forms, the thesis includes a key mathematical element related to the geometrical laws of *n*-fold rotational symmetries. The starting point here is the mathematical concept of *periodicity*. All linearly repeating patterns are periodic, and tilings are one example of such patterns. Tilings can be made using, for example, only rhombuses, regular triangles, squares, or hexagons, whereas it is not possible to make a tiling with only regular pentagons, heptagons, octagons, or any larger regular *n*-sided polygon.

For more than two centuries these mathematical insights in periodicity were used in the study of natural forms in crystallography. Until recently, it was a well-established theoretical and observed fact that all atoms in crystalline solid matter are organized in periodically repeating units. Crystals can have only two-, three-, four-, or six-fold rotational symmetry. According to classic crystallography, repeating five-, seven-, eight-, nine-, or larger *n*-fold rotational symmetries are not possible in the atomic structure of solid matter. However, in 1974 the British physicist Roger Penrose discovered a non-periodic tiling, which possessed many properties of periodic tilings yet had five-fold rotational symmetry, like that of the regular pentagon. The Penrose tiling can be made by using two specific rhombuses. In 1984 the Israeli chemist Dan Shechtman published a remarkable paper announcing the discovery of the first known solid matter which seemed to possess precisely the same bizarre five-fold symmetric structure as the Penrose tiling. Such non-periodic solid matter was soon named "quasicrystals", as their structure was not periodic but rather *quasiperiodic*. Later even more quasicrystalline materials were found with other non-crystallographic rotational symmetries, such as eightfold, tenfold, and twelvefold rotational symmetries.

For two decades I had been studying the geometric properties of tilings with "untypical" rotational symmetries, but nothing serious had come of these studies. When Shechtman received the Nobel Prize in 2011, however, my interest was reawakened. I realized that quasicrystals and quasiperiodic tilings fall nicely within the sphere of my doctoral studies, i.e. basic forms and nature. In March 2012 I discovered a way to construct a quasiperiodic tiling with arbitrarily large *n*-fold rotational symmetry. This solution was first of its kind in mathematics of crystallography. Later I made contact with the mathematician Jarkko Kari, a professor of mathematics from the University of Turku, Finland, who managed to prove – in a strict mathematical sense – my intuitive solution. The discovery and its proof were published in our co-authored paper in the peer-reviewed *Discrete & Computational Geometry*, Vol. 55, Issue 4, June 2016, pp. 972–996. Due to its rather technical nature, the paper is included here as an appendix, but the most essential features of the discovery and its background are explained in the thesis in more accessible terms.

Summary in Finnish

Markus Rissanen

Suomenkielinen tiivistelmä kuvataiteen tohtorityön kirjallisesta osasta

Perusmuodot ja luonto

visuaalisesta yksinkertaisuudesta käsitteelliseen monimutkaisuuteen

Tämä opinnäyte on poikkitieteellinen tutkimus taiteissa ja tieteissä käytetyistä perusmuodoista. Työssä sovelletaan sekä taiteellisia että tieteellisiä lähestymistapoja ja taiteellisten produktioiden lisäksi se koostuu muotojen kulttuurihistoriallisesta ja matemaattisesta tutkimuksesta.

Tutkimusprosessin alkuaikoina keskityin enemmän tieteellisiin visualisointeihin. Olin kiinnostunut kysymyksestä, kuinka annamme näkyvän hahmon sellaisille havaittaville ilmiöille, joita emme pysty jäljittelemällä, eli *mimeettisesti* kuvaamaan. Tutkimuksen esitarkastettuun taiteelliseen osaan kuuluvissa maalauksissani tämä painotus näkyy usein selvästi. Jotkin näistä maalauksista ovat esillä myös tässä kirjallisessa osassa, jossa keskityn enemmän perusmuotojen ja luonnon välisten suhteiden tutkimiseen. Kirjani alkaa klassisen euklidisen geometrian perusmuodoista: ympyrästä, neliöstä ja kolmiosta. Näiden yksinkertaisten hahmojen lisäksi tarkastelen myös haarautuvaa puu-rakennetta perusmuotona. Tämä viimeksi mainittu tarjoaakin sillan perinteiseen euklidiseen geometriaan kuulumattomiin fraktaaleihin, muotoihin joita voimme luonnossa runsain mitoin havaita. Vinoneliöt ovat merkittävässä osassa erityisesti kirjan loppupuolella ja esitetty matemaattinen tutkimustulos perustuu niiden käyttöön. Tästä syystä tarkastelen myös vinoneliötä perusmuotona. Ympyrä, neliö ja kolmio tarjoavat tutkimukselle luontevan lähtö-

kohdan, koska juuri ne muodostavat sen joukon, joka useimmiten tavataan kuvataiteen opetuksen historiassa, erityisesti piirustuksen alkeiden opetuksessa. Tämä kolmikko muodostaa modernistisen ”bauhausilaisen” muotokielen kovan ytimen, ollen samalla platonilaisen häviämättömien ja ikuisten ideoiden käsitteen kuvallinen ruumiillistuma. Vaikka nämä tarkat geometriset muodot ovat hallitsevia vain kulttuurin artefakteissa, niitä löytyy lukematon määriä myös meistä riippumattomasta luonnosta.

Tutkimuksessani esitän kuinka kulttuurimme kuvaa luontoa perusmuotojen avulla kahdella eri tavalla. Näistä kuvaamisen muodoista käytän kahta uutta käsitettä: luonnon esittäminen *havaittujen muotojen* avulla ja luonnon esittäminen *käsitteellisten muotojen* avulla. Ensimmäisen kategorian muodot perustuvat näkyvässä luonnossa joko paljain silmin tai erilaisten instrumenttien avulla havaittaviin muotoihin. Toisen kategorian muodot liittyvät luonnonilmiöiden visualisoimiseen sellaisten keinotekoisilta näyttävien ja usein geometristen muotojen avulla, joita me ihmiset olemme joko vahingossa keksineet tai tarkoituksella konstruoineet kuvataksemme luonnon ilmiöitä tai toimintoja systemaattisesti, teoreettisesti, ”dataorientoidusti”, tai laveasti ilmaistuna: ”tieteellisesti”. Ensimmäinen tapa kuvata luontoa pyrkii osoittamaan miltä luonto *näyttää* kun taas toinen kuvaamisen tapa pyrkii osoittamaan kuinka luonto *toimii*. Edellinen on siis ennen kaikkea tarkoitettu *silmälle* ja jälkimmäinen *mielelle*. Tämä tutkimus myös uudelleenarvioi perusmuodon käsitettä ja esittää vaihtoehtoisen, edellistä laajemman käsitteen *geometristen muotojen käsitteellisistä systeemeistä*. Tällaiset systeemit käyttävät yksinkertaisia geometrisia malleja rakentuen varhaisemmille, yksinkertaisemmille systeemeille. Osoitan kuinka tällaisten käsitteellisten systeemien ketju ulottuu esihistoriallisesta ”proto-geometriasta” modernin fysiikan supersäie-teorioihin.

Maalausten ja kulttuurihistoriallisen muotojen tutkimuksen lisäksi opinnäytteeseen sisältyy eräs keskeinen matemaattisen elementti, joka liittyy *n*:llä jaollisten kiertosymmetrioiden geometrisiin lainalaisuuksiin. Kyseisen osan lähtökohtana on matemaattisen jaksollisuuden eli *periodisuuden* käsite. Kaikki lineaarisesti toistuvat kuviot ovat jaksollisia ja tason *laatoitukset* ovat eräs esimerkki tällaisista kuvioista. Tasopinta voidaan laatoittaa käyttämällä vain esimerkiksi vinoneliöitä tai säännöllisiä kolmioita, neliöitä tai kuusikulmioita, mutta ei käyttämällä ainoastaan 5-, 7-, 8-, 9-, tai suurempia säännöllisiä *n*-kulmioita.

Yli kahden vuosisadan ajan tällaista matemaattisen jaksollisuuden käsitettä käytettiin kristallografiassa kiteiden luonnollisten muotojen tutkimuksessa. Viimeaikoihin asti kaikkien kiinteiden aineiden atomien järjestäytyminen jaksollisesti toistuviin yksikköihin oli sekä teorian että lukemattomien käytännön havaintojen varmistama tosiasia. Kiteillä voi olla vain kahdella, kolmella, neljällä, tai kuudella jaollinen kiertosymmetria. Klassisen kristallografian mukaan toistuva viidellä, seitsemällä, kahdeksalla, yhdeksällä, tai suuremmalla *n*:llä jaollinen kiertosymmetria ei ole mahdollinen kiinteän aineen atomirakenteessa. Brittiläinen fyysikko Roger Penrose löysi kuitenkin 1974 ei-jaksollisen laatoituksen, jolla oli monia jaksollisten laatoitusten ominaisuuksia, vaikka sillä oli säännöllisen viisikulmion kaltainen, siis viidellä jaollinen kiertosymmetria. Penrose-laatoitus voidaan muodostaa käyttämällä kahta erityistä vinoneliötä. Israelilainen kemisti Dan Shechtman julkaisi vuonna 1984 merkittävän artikkelin, jossa ilmoitettiin sellaisen kiinteän aineen löytymisestä, jolla näytti olevan samankaltainen omituinen viidellä jaollinen kiertosymmetrinen rakenne kuin Penrose-laatoituksellakin. Tuollaiselle ei-jaksolliselle kiinteälle aineelle annettiin pian nimi ”quasicrystals” (suom. ”kvasikristallit”), koska sellaisen atomirakenne ei ollut jaksollinen, eli *periodinen*, vaan *kvasiperiodinen*. Myöhemmin löydettiin vielä muitakin kvasikristallisia aineita, joilla oli toisia ei-kristallografisia, kuten kahdeksalla, kymmenellä, tai kahdellatoista jaollisia kiertosymmetrioita.

Yli kahden vuosikymmenen ajan tutkin itsekseni eräiden ”epätyypillisiä” kiertosymmetrioita omaavien laatoitusten geometrisia ominaisuuksia, vaikkei näistä tutkimuksista mitään vakavasti otattavaa näyttänytkään syntyvän. Kiinnostukseni aiheeseen heräsi kuitenkin uudella tavalla, kun Shechtman sai Nobel-palkinnon 2011. Ymmärsin, kuinka kvasikristallit ja kvasiperiodiset laatoitukset sopivatkin mukavasti tohtoriopintojeni aihepiiriin, siis perusmuotojen ja luonnon suhteeseen. Maaliskuussa 2012 löysin mallin, jonka avulla voidaan konstruoida kvasiperiodinen laatoitus mille tahansa, siis myös kuinka suurelle tahansa, *n*:llä jaolliselle kiertosymmetrialle. Tämä ratkaisu oli ensimmäinen laatuaan matematiikan tai kristallografian alalla. Myöhemmin sain kontaktin matemaatikko Jarkko Kariin, Turun yliopiston matematiikan professoriin, jonka onnistui todistaa – matemaattisen tiukassa merkityksessä – intuitiivinen ratkaisuni. Löydös ja sen todistus julkaistiin yhteisartikkelina vertaisarvioidussa lehdessä *Discrete & Computational Geometry*, Vol. 55, Issue 4, June 2016, pp. 972–996. Kyseisen artikkelin melko teknisestä luonteesta johtuen se on mukana tässä kirjassa liitteenä, mutta ratkaisun keskeinen periaate ja tausta esitetään myös helpommin lähestyttävässä muodossa.

Appendix A

Hex Rosa

Markus Rissanen

Roses from Penrose

The famous Penrose tiling[1] was the original inspiration for the system of rhombic tilings described in this paper. Nevertheless, the properties of this system are much simpler than those of the Penrose tiling. Unlike the Penrose tilings, done with two rhombuses with specific matching rules which guarantee that they are aperiodic[2], this system uses rhombuses with no matching rules; thus, no tiling described here is aperiodic. However, except for values n = 3, 4, and 6 all the tilings described are nonperiodic.

Figure A1 shows some patches for the first few values of n. As these patches greatly resemble flowers with their petals, I call them *roses*. The term *rosette* is also used for similar types of rhombic compositions, but it is also used as a patch with exactly the same rhombuses inside an identical regular perimeter but in any possible order.[3] Thus, I use here the word "rose" to describe patches seen in **Fig. A1,** with

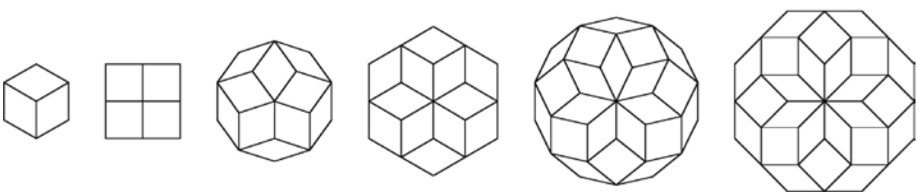

Figure A1 The roses for n = 3, 4, 5, 6, 7, 8.

[1] Martin Gardner, Extraordinary nonperiodic tiling that enriches the theory of tiles, *Scientific American* 236 (1977), pp. 110–121

[2] Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, (1987), pp. 519–548

[3] Alan H. Schoen at http://schoengeometry.com/b-fintil.html (accessed 2016-04-14)

both rotational and reflectional symmetry. The 5-fold symmetric roses are easily seen in all rhombic Penrose tilings; see, for example, Fig. 10.3.20, (pp. 544–545) in Grünbaum and Shephard (1987).

Rose patterns in **Fig. A1** can be seen as two-dimensional projections of three-dimensional polar zonohedrons.[4] It is even difficult *not* to see some illusory three-dimensionality in these patterns, especially as the leftmost pattern in **Fig. A1** is a typical way of depicting a cube. These patches also emerge in some two-dimensional projections taken from even higher *n*-dimensional hyper-cubic lattices.[5] The results described in this paper are only a summary of about 25 years' of intermittent studies based on a few simple ideas and observations. More serious theoretical results were later developed from the tilings described here, and they have also been published recently.[6]

Hexagonal Modules

In Figure 2, one can see a multitude of roses repeating seemingly *ad infinitum* in all directions. Actually, this is the case, and corresponding tilings can be constructed for all $n \geq 3$. This is achieved using a certain repeating hexagonal module, shown with grey lines in **Fig. A2.** I call this hexagonal module *delta hexagon*.

The delta hexagon has all sides of the same length and all opposite sides parallel. One main property of the delta hexagon is defined by an integer k, which divides 360° in k equal parts. The interior angles of the delta hexagon are $360°/k$ in two opposite corners and $180°((k–1)/k)$ in four other corners, the total sum of interior angles always being 720°.

In May 1999 I made the simple observation that a plane can be tiled with these hexagons for all $k \geq 2$ in the following manner. First we place k delta hexagons tightly around a point in such a manner that all of these hexagons share sides with their nearest neighbours. We continue the arrangement by placing k congruent

[4] B. L. Chilton and H. S. M. Coxeter, "Polar Zonohedra", *Amer. Math. Monthly* 70, (1963), pp. 946-951.

[5] Whittaker, E. J. and Whittaker, R. M., "Some Generalized Penrose Patterns from Projections of *n*-Dimensional Lattices", *Acta Crystallographica*, Vol. A44, Part 2, (1988), pp. 105–112

[6] Jarkko Kari and Markus Rissanen, "Sub Rosa, a System of Quasiperiodic Rhombic Substitution Tilings with *n*-Fold Rotational Symmetry", *Discrete & Computational Geometry*, Vol. 55, Issue 4, June 2016, pp 972-996, the peer-reviewed paper is also available at http://link.springer.com/article/10.1007/s00454-016-9779-1 (2016-04-04) and a pre-review version is also available online at http://arxiv.org/abs/1512.01402 (2015-12-04).

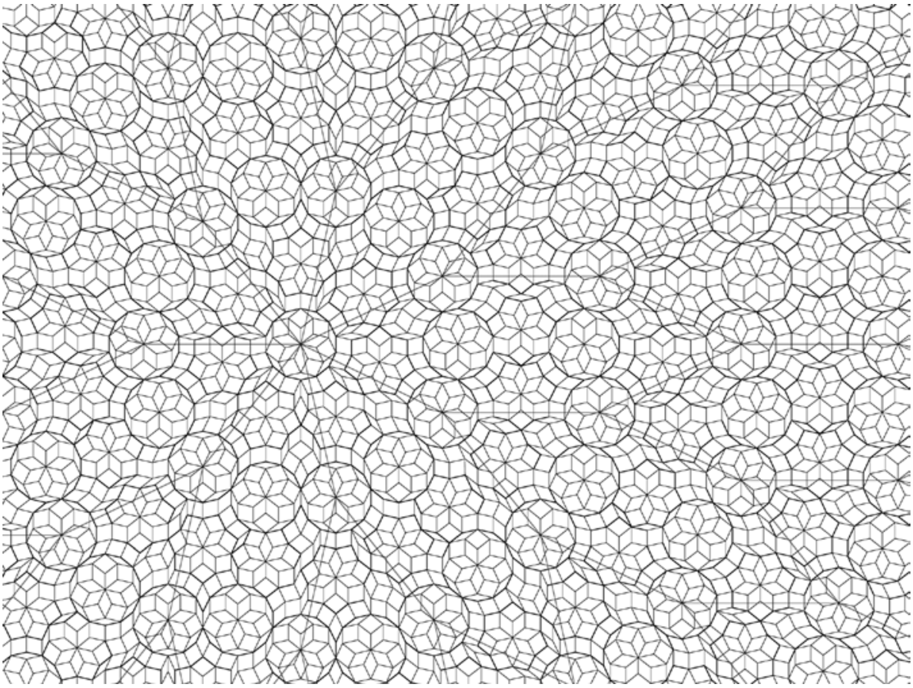

Figure A2 The Hex Rosa tiling for n = 7 with its hexagonal modules also shown.

hexagons between each two previous hexagons, between them we place 2k hexagons, and between them we again place 2k hexagons, and so on with 3k, 4k, 5k, etc. hexagons. This produces a pattern made of delta hexagons as seen in Figure 3. Compare **Figs. A2 and A3** for n = 7; note that these two images are drawn in different scales.

Another, simpler way of tiling the plane with hexagons is the periodic "chicken-wire" pattern. In **Fig. A3,** we see that there are 14 (or generally 2k) areas, or "wedges", which actually consist of the "chicken-wire" pattern. The borders of neighbouring "wedges" have a zigzag form depicted with darker lines in **Fig. A3.** Delta hexagons are called such because triangularly-growing wedges made of them slightly resemble a river delta , see **Fig. A3.**

Only k wedges meet in the centre, and these wedges separate into other k wedges. See **Fig. A3** and **Fig. A4** for four corresponding delta hexagon wedges for k = 2 (left) and six wedges for k = 3 (right). The hexagons are always connected only edge-to-edge. Note that for k = 2, the delta hexagon with its six vertices of 90°, 90°,

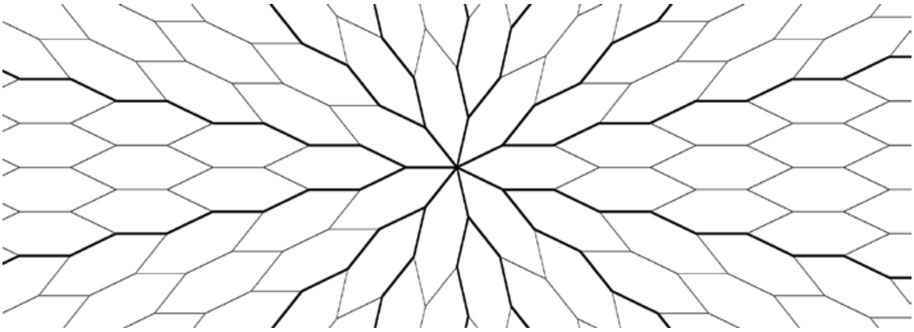

Figure A3 Fourteen delta hexagon wedges for $k = 7$. Half of them meet in the centre.

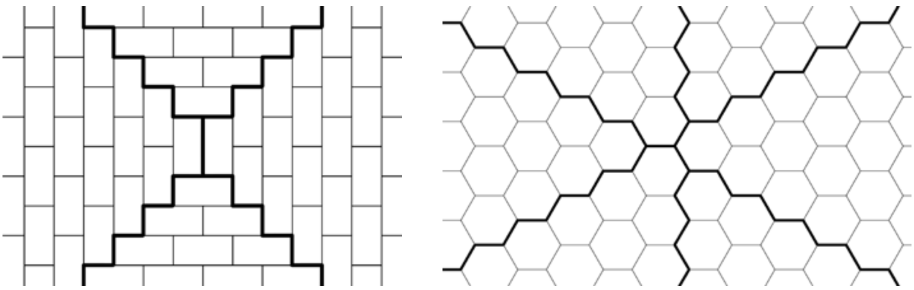

Figure A4 Four delta hexagon wedges for $k = 2$ (left), and six delta hexagon for $k = 3$ (right).

180°, 90°, 90°, 180° is also a proper rectangle, with four vertices of 90°.

The nonperiodic k-fold rotationally symmetric delta hexagon pattern and the more typical "chicken wire pattern" are by no means the only possible tilings with delta hexagons; actually, there are infinitely many non-congruent tilings possible with delta hexagons, but I will not go into them in this paper.

Filling the Modules

Ultimately, the delta hexagons are treated here only as provisional modules in order to obtain rhombic tilings with a fairly high concentration of locally n-fold symmetric roses for all $n \geq 3$. As this system uses hexagons and rose patterns, it is called *Hex Rosa*. Such a tiling is obtained here by placing a rose at every vertex of a delta hexagon. This guarantees a tiling with an even and nonperiodic distribution of an infinite number of locally n-fold rotationally symmetric patches plus one centre of globally n-fold rotational symmetry.

A Hex Rosa tiling can be constructed for all $n \geq 3$. In **Fig. A2,** we see that there are actually two types of roses – convex and concave – in a Hex Rosa tiling. First, there are the convex or "closed" roses, whose perimeter is in the shape of a regular polygon, as seen in **Fig. A1.** Secondly, there are the concave or "open" roses in between the convex ones. Peeling off the outermost "petals" from the closed roses gives corresponding open roses for all $n \geq 5$. For $n = 3$ and 4, the open roses are mere thin sticks with no area. In addition to these new concave roses and convex roses positioned in the vertices, there are two more convex roses inside the hexagon; see **Figures A5** (right) and **A6.**

A distinction between odd and even values of n needs to be made. The perimeters of the roses are regular $2n$-gons of (with edges of one unit) for odd values and regular n-gons (with edges of two units) for even values. The rotational symmetry of a rose is n-fold for all n, odd and even; see **Fig. A1.** Every n belongs to one of the following three cases, which characterize the Hex Rosa tilings:

(1) If n is odd, there are $(n–1)/2$ different prototiles, or elementary rhombuses: $(1, n–1)$, $(2, n–2)$, … $((n–1)/2, (n+1)/2)$, written as multiples of the angle $180°/n$.

(2) If n is even and $n/2$ is odd, there are $(n–2)/4$ different elementary rhombuses: $(1, (n–2)/2)$, $(2, (n–4)/2)$, …$((n–2)/4, (n+2)/4)$, written as multiples of the angle $360°/n$.

(3) If n is even and $n/2$ is also even, there are $n/4$ different elementary rhombuses: $(1, (n–2)/2)$, $(2, (n–4)/2)$, … $(n/4, n/4)$, written as multiples of the angle $360°/n$. Note that only in this third case (3), there are squares among the rhombuses.

The differences between odd and even values emerge while defining the angles of the delta hexagons. For odd values of n, we can identify simply $k = n$; thus, the angles of the delta hexagon are $360°/n$ in two opposite vertices and $180°((n–1)/n)$ in four other vertices. For odd values, there are n wedges meeting in the centre with another n wedges positioned tightly in between them; see **Fig. A4** (right).

For even values of n, the identity $k = n$ would cause unsolvable problems later while tiling the delta hexagons with rhombuses. To avoid this problem, we define $k = n/2$ for all even values of n. Thus, for even n, the angles of the delta hexagon are $720°/n$ in two opposite vertices and $180°((n–2)/n)$ in four other vertices. For even n, there are $n/2$ wedges meeting in the centre with another $n/2$ wedges positioned tightly in between them, see **Fig. A4** (left).

The Rose Gardens

I call the delta hexagon, which contains roses in its corners and which is legitimately tiled with rhombuses, a *rose garden*. Note that only a part of a rose in the corner is contained inside the specific hexagon, and the rest of it belongs to neighbouring hexagons. As the sum of the interior angles of a hexagon is always two full circles, the corners of one hexagon contain rhombuses for only two complete roses.

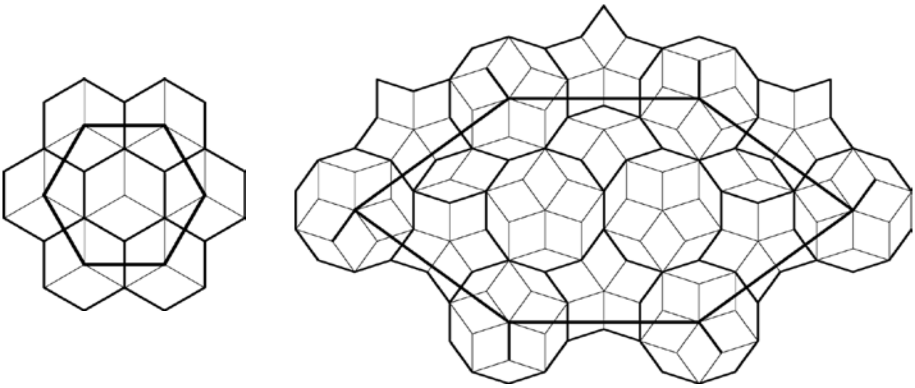

Figure A5 The rose garden for $n = 3$ (left), and the rose garden for $n = 5$ (right).

For $n = 3$, all roses can be oriented the same way; see **Fig. A5** (left). For $n > 3$, a specific orientation rule is needed to ensure that roses orientate correctly with the neighbouring rose gardens. The orientation rule is depicted in **Figs A5** (right) and A6: the long edge of the hexagon and one short line coming from the centre of a rose have to meet perpendicularly outside the hexagon, and together they form a chain of six connected super-slim L-shapes defining the edges of the rose garden. When rose gardens are legitimately connected, that is edge-to-edge, the roses at the corners are cordially shared by all gardens meeting at the point in all possible combinations. I leave it to the reader to assure herself of this fact.

For clarity's sake, an excessive number of rhombuses are depicted outside the edges of the hexagons in **Figs. A5** and **A6.** Eventually, the edges of a rose garden have to be "cut clean" to avoid overlapping with the rhombuses of the neighbouring hexagons, but to emphasize the number, position, and the shapes of the roses and rhombuses, the sides of the gardens have been left "uncut" in the following images. To complete the Hex Rosa tiling for a plane, it suffices to find a proper way to tile the finite interior of a rose garden as its delta hexagon shape allows the tilings of an infinite plane in a way seen in previous **Figs. A3** and **A4.**

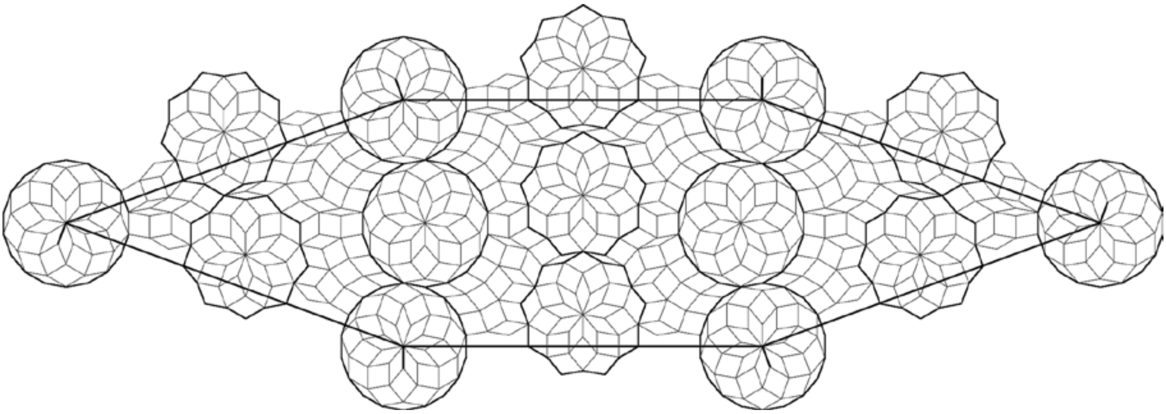

Figure A6 The rose garden for n = 9.

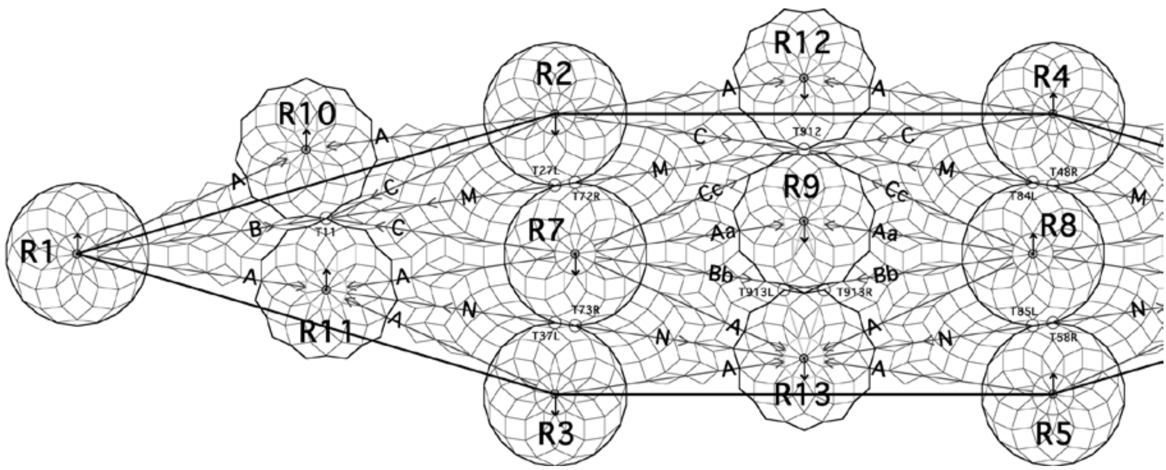

Figure A7 The map of a general rose garden with its paths.

Paths to the General Solution

In the following, I limit my presentation to the odd values of n. As n grows, one must recognize certain rules and regularities in order to systematically construct new rose gardens for the larger values of n. Alternative solutions and variations naturally exist. I prefer a system where the rhombuses are arranged in such a way that straight lines from one rose centre to another rose centre will bisect all rhombuses along the line. I call these lines *paths*.

Fig. A7 shows how these paths are located in a "general" rose garden with the value n = 11 used in this particular image. The convex or "closed" roses are numbered R1, … R8, whereas the concave or "open" roses are numbered R9, … R15. Roses R1, … R6 occupy the corners. Due to the lack of space, convex rose R6 in the rightmost edge is cut out from the image as well as the two concave roses, R14 and R15, from the same direction. I trust that the reader is able to imagine the missing right-hand part of **Fig. A7.**

The size of the hexagon in relation to the size of the rose is defined in the following way. Lines R1-R2 and R1-R3 form an angle of 360°/n. There is a rose (R7) in between roses R2 and R3, both of which share one edge with R7. The distances R1-R2, R1-R3, and R1-R7 are equal, as are the angles R2-R1-R7 and R7-R1-R3; please compare **Figs. A5, A6,** and **A7.** There are different types of paths, marked with letters A, B, C, M, N, Aa, Bb and Cc in **Fig. A7.** In addition to paths running from centre to centre, there are paths which either start or end at "tangent" points where some roses touch each other. These points are marked with a small circle and T(xxx) in **Fig. A7.** If we define the edge of the rhombuses as one unit of length, we end up with the following equations:

- The outer radius of the rose, that is the radius of the circumscribed circle $R(n) = 1/(2\sin(w))$
- The length of the side of the rose garden hexagonal $S(n) = \cos(w)/2\sin^2(w)$
- The length of the paths $P(n) = 1/(4\sin^2(w))$, where, in all three equations, $(w) = 90°/n$

$P(n)$ is the length of "full" paths A, B, C, M, and N. Path Aa is one rhombus shorter in length. Paths Cc and Bb are of full length, but exceptionally do not bisect their last rhombuses. There is an elegant connection between the length of the paths and the radii of the roses: $P = R^2$.

Path A can be considered as the "archetype" for all these paths. The composition of paths B and C relate more closely to A than the composition of paths M and N. The composition of a path can be coded with the bisected rhombuses.

In **Fig. A7,** for example, path A from R1 to R10 reads as (0-4-8)-(0-2-4)-0-(6-2) and path A from R2 to R10 reads as (2-6-10)-(0-2-4)-0-(6-2); please note the reading direction of paths, marked with small arrows. The underlined values refer to the rhombuses inside the roses. In **Fig. A7,** path B reads as (2-6-10)-(0-2-4-6)-(0-2), which has exactly the same rhombuses as path A from R2 to R10, but in another order. All full paths are of the same length, and they contain the same rhombuses with the possible exception of the underlined part, which runs inside a rose. Depending on its direction, the radius R(n) of every rose is either composed of rhombuses (0-4-8…) or (2-6-10…); compare, for example, rhombuses inside R1 along paths A and B.

Unfortunately, I am not able to give an extensive description of the compositions and properties of the paths in this paper. I will just mention the following observations. It is rather straightforward how the complete tables expressed in rhombuses are formed for all paths A, B, C, M, N, Aa, Bb, and Cc for all odd *n*. For example, for *n* = 19, path A has the rhombuses (0-4-8-12-16)-(0-2-4-6-8-10-12)-(0-2-4-6-8)-(0-2-4)-(0)-(14-10-6-2), and path M has the rhombuses (***0***)-(***4***-2-0)-(***8***-*6*-4-2-0)-(***12***-*10*-8-6-4-2-0)-(***16***-*14*-12-10-8-6-4-2-0), where the rhombuses can be re-arranged to obtain the composition of path A. For example, the first rhombuses marked with ***bold italics (0, 4, 8, 12, 16)*** form the left underlined rose-part of the A, that is (0-4-8-12-16). and the second rhombuses marked with *normal italics* (*2, 6, 10, 14*) form the mirror sequence of the right underlined rose-part of path A, that is, (14-10-6-2).

Conclusions of the Hex Rosa

Paths paved with bisected rhombuses give a robust framework for tiling the complete rose garden for all *n*. In addition to these straight lines, each type of path defines a "natural" tiling in its vicinity. Roses, paths and their surroundings combined cover the hexagonal rose garden almost completely. For larger *n*, there are some small areas which are left in between these three systematically coverable types of areas. Nevertheless, I believe all such small areas can always be tiled ad hoc before further analysis reveals some complete rules for them as well.

Beyond the Hex Rosa

At the end of 2011 the paths depicted in **Fig. A7** gave me an idea about transforming the Hex Rosa tilings into substitution tilings. Just as the actual tiling in the Hex Rosa consists of rhombuses, some paths also define larger rhombuses with the same angles, for example, paths A-A-B-C between R1 and R2, paths C-C-C-Cc between R2 and R7, and Cc-Cc-A-A between R7 and R8. This observation raised a natural question: if the unit rhombuses can be arranged to form such larger "super-rhombuses", would it not be possible to arrange these larger "super-rhombuses" into even larger "super-super-rhombuses"? Or, if this was possible, wouldn't it mean that the unit rhombuses could be seen as consisting of smaller "micro-rhombuses" and so on, *ad infinitum*, in both directions, zooming in and zooming out?

Such a substitution was not achieved easily, but eventually, a viable solution was found. With the substitution process, the hexagonal module was discarded, and the Hex Rosa evolved into Sub Rosa. It turned out that the Sub Rosa tilings are even linearly recurrent or quasiperiodic, i.e. for every finite patch, there exists infinitely many identical copies of it within some finite distance, which is linearly dependent on the diameter of the selected patch. Together with the nonperiodic rotational symmetry for all *n* and the crystallographic restriction theorem, which forbids more than one global centre of rotational symmetry for values other than n = 2, 3, 4, or 6, there is a seemingly paradoxical situation within such tilings. They must have one absolutely unique centre of global rotation, and yet their quasiperiodicity guarantees that one is also able to take as large a (finite) area as wished around this point, and there exist an infinite number of congruent patterns within some finite distances from each other.

The Penrose tiling was the first tiling with such a quasiperiodic property for a value ($n = 5$) other than the periodic ones, $n = 2$, 3, 4, and 6. As the Sub Rosa proves similar properties to exist for all $n \geq 2$, I conclude by saying that the circle from my original inspiration, the Penrose tiling, has nicely closed.

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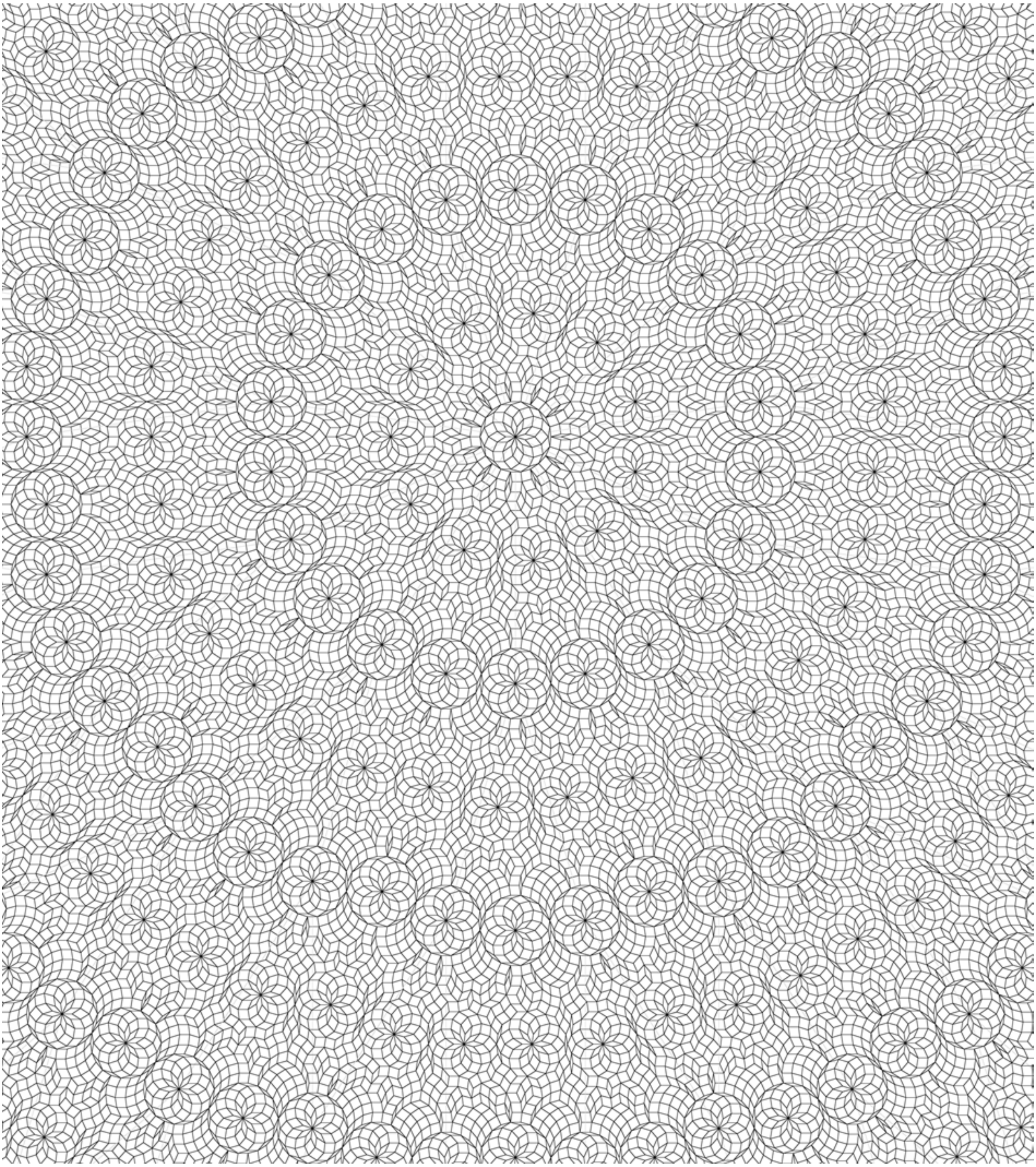

Figure A8 The Hex Rosa tiling for n = 11

Appendix B

Sub Rosa, A System of Quasiperiodic Rhombic Substitution Tilings with n-Fold Rotational Symmetry

Jarkko Kari & Markus Rissanen

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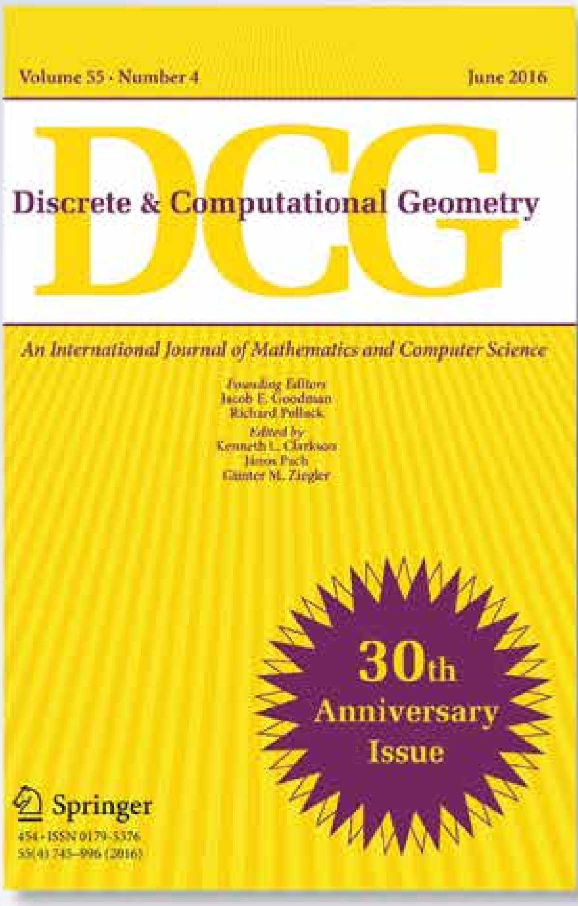

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Sub Rosa, A System of Quasiperiodic Rhombic Substitution Tilings with n-Fold Rotational Symmetry

Jarkko Kari[1] · **Markus Rissanen**[2]

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Abstract In this paper we prove the existence of quasiperiodic rhombic substitution tilings with $2n$-fold rotational symmetry, for any n. The tilings are edge-to-edge and use $\lfloor \frac{n}{2} \rfloor$ rhombic prototiles with unit length sides. We explicitly describe the substitution rule for the edges of the rhombuses, and prove the existence of the corresponding tile substitutions by proving that the interior can be tiled consistently with the given edge substitutions.

Keywords Substitution tiling · Quasiperiodic · Rotation symmetry · Rhombic tiling

1 Introduction

A tiling is a covering of the infinite plane using copies of a finite number of different prototiles, without leaving gaps or letting the tiles overlap. In this paper we consider edge-to-edge substitution tilings of the plane with rhombuses whose edges are of unit length. The edge-to-edge condition means that two tiles in the tiling are either disjoint, have one common vertex or have one common edge. A substitution rule tells how to replace enlarged tiles by patches of tiles. By iterating the process of inflating the tiles and substituting the corresponding patches, one obtains a sequence whose

Editor in Charge: Kenneth Clarkson

Jarkko Kari
jkari@utu.fi

Markus Rissanen
markus.rissanen@gmail.com

1 Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland

2 Töölönkatu 11 A 130, 00100 Helsinki, Finland

limit is a tiling of the infinite plane. Such substitutions are a popular way to produce non-periodic tilings. The Penrose tilings [4,9] are the best known examples of this effect.

We say that a tiling has n-fold *rotational symmetry* (with center P) if the tiling is invariant under the rotation around P by the angle $\frac{2\pi}{n}$. Note that this requires a perfect global symmetry of the entire tiling. There may exist an infinite number of centers of rotation for n-fold rotations only if $n = 2, 3, 4$ or 6. For all other values there can be at most one central point of rotational symmetry. This fact is known as the *crystallographic restriction*.

All our tilings are by rhombic prototiles with unit length sides. As usual we shall define the substitution process in two steps. A tile is first enlarged by a constant called the *scaling factor*. The enlarged tile is then replaced by a patch of prototiles that cover the same area as the enlarged tile does. This second step is called the *substitution rule*. The boundaries of this new replaced tile do not have to go along the boundaries of the merely enlarged tile but their areas have to be equal. If there are some kind of "dips" inwards in the perimeter of the replaced tile also corresponding "dips" outwards have to exist and in the right places, so that no overlapping occurs along the edge of two neighbouring enlarged tiles. This condition can be guaranteed by an associated *edge substitution rule*.

Starting with any tile, the substitution process may be iterated to obtain higher order images. The substitution is *primitive* if there exists k such that the k'th order image of every tile contains a copy of every prototile. All our substitutions will be primitive with $k = 1$, so that each tile appears in the image of each tile.

A *substitution tiling* is any non-overlapping covering of the plane by the tiles such that every finite patch of tiles present in the tiling appears in some higher order image of some tile. Such patches are called *legal*. In practice, we generate tilings by iterating the substitution from an initial patch of tiles, positioning the obtained patches so that each generation properly contains the previous one in its interior, and take the limit of the process. If the initial patch is such that it appears in a higher order image of some tile then the obtained limit is clearly a substitution tiling.

A tiling is *recurrent* if every finite patch of tiles that appears in the tiling appears infinitely many times in it. It is *uniformly recurrent* (or *quasiperiodic*, or *repetitive* [1]) if every finite patch that appears somewhere in the tiling appears within distance D of every point of the plane, for some D. The value D may depend on the patch. If D can be bounded by a linear function of the diameter of the patch then the tiling is *linearly recurrent*, or *linearly repetitive* [1]. Tilings generated by primitive substitutions are automatically uniformly recurrent. Indeed, any finite patch that appears in a tiling is legal, i.e., appears in some higher order image of some tile, and by primitivity then, in the m'th order image of every tile, for some m. Radius D can be taken to be the maximum diameter of the m'th order images of the tiles. A more careful analysis reveals that our tilings are even linearly recurrent.

In this paper we are considering the problem of finding a primitive substitution that generates a tiling with n-fold rotational symmetry. Values $n = 2, 3, 4$ and 6 are trivial, and there are even periodic solutions: The regular square tiling for $n = 4$ and the case $n = 6$ are considered below as the first member of the SUB ROSA family. The famous Penrose rhombuses provide a solution to the case $n = 5$ [4,5,9], and the Ammann–

Beenker tiling for $n = 8$ [2]. Recently, in [3] a computer algorithm was described to search for solutions for arbitrary n. Many substitution rules were discovered for $n = 5$ and $n = 7$, but the computational complexity of the problem was reported to be prohibitive already in the case $n = 11$.

We are not aware of known solutions for general n. In [6] primitive rhombic substitutions were provided for all n that can be iterated on an n-fold rotationally symmetric initial patch to obtain in the limit a tiling with n-fold symmetry. However, the initial patch does not appear in any higher order image of any tile, so the obtained tiling is "singular" and not a substitution tiling according to the stricter definition of the present paper. Moreover, the tiling is not recurrent, since the initial patch only appears once in the tiling. In [6] the term *non-singular substitution tiling* was used to describe the type of substitution tilings used in the present paper.

The main result is the following theorem.

Theorem 1 *For every n, there exists a quasiperiodic rhombic substitution tiling with* $2n$*-fold rotational symmetry.*

We start by setting the notations in Sect. 2. The discussion is then divided in two parts, depending on whether n is odd or even. In Sect. 3 we show, as examples, our substitutions for small odd n, and discuss the scaling factors and the edge substitution rules for arbitrary odd n. Section 4 provides the analogous discussion for even n. In Sect. 5 we provide the main proof. We explicitly describe the edge substitution rule, and express the boundary of the enlarged rhombus as a circular word of unit vectors. In Sect. 5.1 we develop a rewrite system on the boundary word to check the tileability of the interior. In Sect. 5.2 the notation is set up for the complete case analysis that we do in Sect. 5.3.

2 Notations

We use only rhombic tiles. Let $n \geq 2$ be a fixed integer. For any positive integers x and y that satisfy $x + y = n$, we denote by pair (x, y) the rhombus with unit length edges, and angles $\frac{x\pi}{n}$ and $\frac{y\pi}{n}$. Pairs (x, y) and (y, x) are the same shape. These $\lfloor \frac{n}{2} \rfloor$ *unit rhombuses* are used in our $2n$-fold symmetric solutions. We label the corners of the (x, y) rhombus by x and y. In a valid edge-to-edge tiling then, the labels of the corners meeting at a vertex have to sum up to $2n$. The patch of tiles that is an image of a rhombus under our substitution is called a *super-rhombus*.

The system described here utilizes a rotational and reflection symmetric simple arrangement of rhombuses around a single point. We present here two such patterns: The first one has n-fold rotational symmetry, the second one $2n$-fold symmetry.

First we construct the smaller pattern where we assume $n \geq 3$. We place n copies of $(2, n-2)$-rhombuses around a point with their $\frac{2\pi}{n}$ vertices meeting at the point. This pattern is surrounded by a ring of n copies of $(4, n-4)$-rhombuses. These properly match at the corners since $(n-2) + (n-2) + 4$ equals $2n$. This pattern is in turn surrounded by a ring of $(6, n-6)$ rhombuses, then by $(8, n-8)$ rhombuses, and so on. This procedure is repeated until the outmost ring of $(n-1, 1)$ or $(n-2, 2)$ is reached, depending on whether n is odd or even. On each step, the angles add up properly at

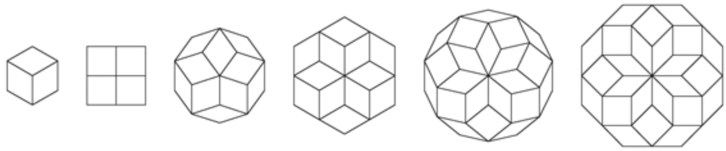

Fig. 1 Rose R_1 for $n = 3, 4, 5, 6, 7, 8$

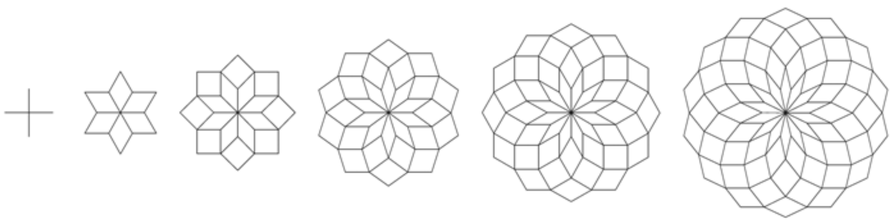

Fig. 2 Rose R_2^1 for $n = 2, 3, 4, 5, 6, 7$

each vertex: on the k'th round the four meeting angles have labels $2(k-1)$, $n-2k$, $n-2k$ and $2(k+1)$, and these add up to $2n$. Note that this analysis also holds on the first round $k = 1$, with the interpretation that the process is initialized with *n zero rhombuses*, i.e., mere edges.

The boundary of the obtained pattern is a regular $2n$-polygon with edges of length 1, if n is odd, and a regular n-polygon with edges of length 2 when n is even. The patterns for $n = 3, 4, \dots, 8$ are shown in Fig. 1.

Secondly we construct the larger pattern. Here $n \geq 2$. We place $2n$ copies of $(1, n-1)$-rhombuses around a point with their $\frac{\pi}{n}$ vertices meeting at the point. This pattern is surrounded by a ring of $2n$ copies of $(2, n-2)$-rhombuses. Which in turn is surrounded by a ring of $2n$ copies of $(3, n-3)$-rhombuses. This procedure is repeated until we construct the outmost ring using $(n-1, 1)$ rhombuses. The result is a regular $2n$-polygon with edges of length 2. Again, it is straightforward to verify that the angles add up properly at each vertex. Also here we can imagine the zeroth round to consist of $2n$ zero rhombuses (= unit edges) around the central point.

Frequently we leave out a number of outer rings. For example, Fig. 2 shows patterns for $n = 2, 3, \dots, 7$ where the last ring of $(n-1, 1)$-rhombuses is omitted. Note that in the case $n = 2$ the omission of the last ring leaves only the degenerate zeroth round.

These patterns greatly resemble a flower with its petals and therefore we call them *roses* and denote them by R_a^b, where $a \in \{1, 2\}$ denotes the type of the rose, and $b \in \mathbb{N}$ denotes number of missing rings. That is, the patterns shown in Fig. 1 are R_1^0, or simply R_1, and patterns in Fig. 2 are R_2^1. Roses R_1^b have n-fold rotational symmetry while roses R_2^b have $2n$-fold rotational symmetry. Note that R_1 for even n only uses tiles (x, y) with even x and y. In fact, R_2 for any n is identical to R_1 for $2n$. For $n \geq 6$ it is possible to construct even larger, a third type of rose-pattern with $2n$-fold rotational symmetry. Examples of such larger type are in Figs. 6 and 8, and if completed, the result is a regular $2n$-polygon with edges of length 4. However, this, or R_1, or any another possible new type of rose is not necessary to prove our main theorem. Instead, roses R_2^1 play a central role in the constructions.

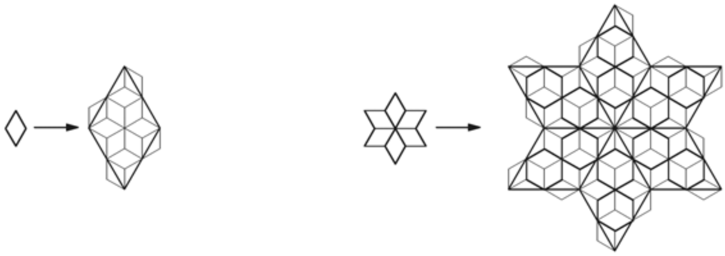

Fig. 3 Substitution rule for $n = 3$, and the rose R_2^1 for $n = 3$ after one substitution

The tilings discussed in this paper use substitutions and rose like patterns, the system is hence called SUB ROSA. We first consider separately the cases of odd n and even n, starting with the odd cases.

3 Scaling Factors and Substitution Rules for Odd n

For $n \in \mathbb{N} \setminus \{0\}$ we define $S(n)$ to be the scaling factor of the substitution. For odd n,

$$S(n) = \cos\left(\frac{\pi}{2n}\right) / \sin^2\left(\frac{\pi}{2n}\right). \tag{1}$$

3.1 Substitution Rule for $n = 3$

For $n = 3$ the SUB ROSA tiling is periodic. For the sake of completeness we will first examine the tiling for value $n = 3$ before moving to the non-periodic tilings starting with $n = 5$. For $n = 3$ there is only one rhombus in the tile set, namely $(1, 2)$. The substitution rule is shown on the left in Fig. 3. The enlarged rhombus is depicted with thick edges and is replaced by 12 rhombuses $(1, 2)$. The edges of the enlarged rhombus bisect two unit rhombuses out of which one is counted in while the other one is counted out. The resulting tiling after one step of substitution on the starting pattern R_2^1 is seen on the right in Fig. 3.

The following aspects are easy to see in the case $n = 3$, but they also hold for the larger values of odd n.

- *The edge substitution rule* All edges of enlarged rhombuses bisect an identical sequence of unit rhombuses. The sequence has even length and it is mirror symmetric. If we represent each bisected unit rhombus as the label of its bisected angle, the edge substitution rule can be written down as the sequence $\Sigma(n)$ of these labels. For example, the edge substitution rule for the case $n = 3$ is represented by the sequence $\Sigma(3) = 1, 1$ since the edge of an enlarged rhombus cuts two unit rhombuses along their long diagonals, bisecting their angles of label 1. The edge substitution rules for larger values of odd n are shown in Table 1.
- The edge substitution rule above guarantees that the super-rhombuses match each other without gaps or overlaps, so that tile substitutions are consistent. We simply include in the super-rhombus half of the unit rhombuses that are bisected by its

edge. More precisely, when following the edges of the enlarged rhombus clockwise, we count in the first half of the bisected unit tiles of each edge and count out the second half. In this way each unit rhombus gets included exactly once since each orientation of an edge between two rhombuses is clockwise for one of the incident rhombuses and counterclockwise for the other.

- In an edge-to-edge tiling by super-rhombuses there are roses R_2^1 centered at all corners of the super-rhombuses. In other words, a super-rhombus has a sector of R_2^1 centered at each corner, and these four sectors together form the full R_2^1 rose. Combined with the fact that there is at least one unit rhombus vertex completely in the interior of each super-rhombus, this implies that the second order image of each tile contains pattern R_2^1. Also, since R_2^1 contains a copy of every prototile in our system, the patch substituted for each tile contains a copy of every prototile and the substitution is then primitive.
- We iterate the substitution from the starting pattern R_2^1. By the point above, the image of R_2^1 has R_2^1 at its center, so we can align the centers of consecutive generations and take the limit to obtain a substitution tiling, Sub Rosa. Strictly speaking, for odd n, the central rose R_2^1 quarter turns in each generation. The roses at even generations and odd generations properly align with each other, but between consecutive generations a quarter turn is required. The generated tiling is a fixed point of the second iterate of the substitution.
- In all our drawings the depicted unit rhombuses, in fact, come with an isometry φ that maps a rhombic prototile T into the given position. Because the rhombuses have the dihedral D_2 symmetry group (D_4 in the case of a square), the image $\varphi(T)$ alone does not contain the full information about φ. To identify φ uniquely, one may consider the unit rhombuses in our illustrations *oriented*. All tiles are positively oriented, i.e., they come with an even isometry. The orientations in the starting pattern R_2^1 are invariant under the $2n$-fold rotational symmetry of the pattern, and the sectors of R_2^1 at the corners of super-rhombuses are oriented correspondingly to guarantee the proper orientations in the roses R_2^1 that are formed around the vertices of the super-rhombuses.
  Note that the orientation of tiles becomes irrelevant if the super-rhombuses have the same symmetries as the corresponding unit rhombus: in this case the substitution is deterministic even without knowing the orientations. We argue in Remark 2 in Sect. 5.1 that our substitutions can always be made with the necessary symmetries.
- We always iterate Sub Rosa from the start pattern R_2^1. As patch R_2^1 has $2n$-fold rotational symmetry, the obtained tiling has that symmetry as well. As the substitution is primitive and the start pattern R_2^1 appears in the second order image of each tile, the tiling is uniformly recurrent.

3.2 Substitution Rules for $n = 5$

For $n = 5$ the tile set consists of two rhombuses, $(1, 4)$ and $(2, 3)$. The substitution rule is shown in Fig. 4. In the figure each integer label represents a multiple of $\frac{\pi}{5}$. It is easy to see that each meeting point of vertices sums up to 10 or in other words, is a full circle 2π. All six points listed above for $n = 3$ apply also here, as they do for all

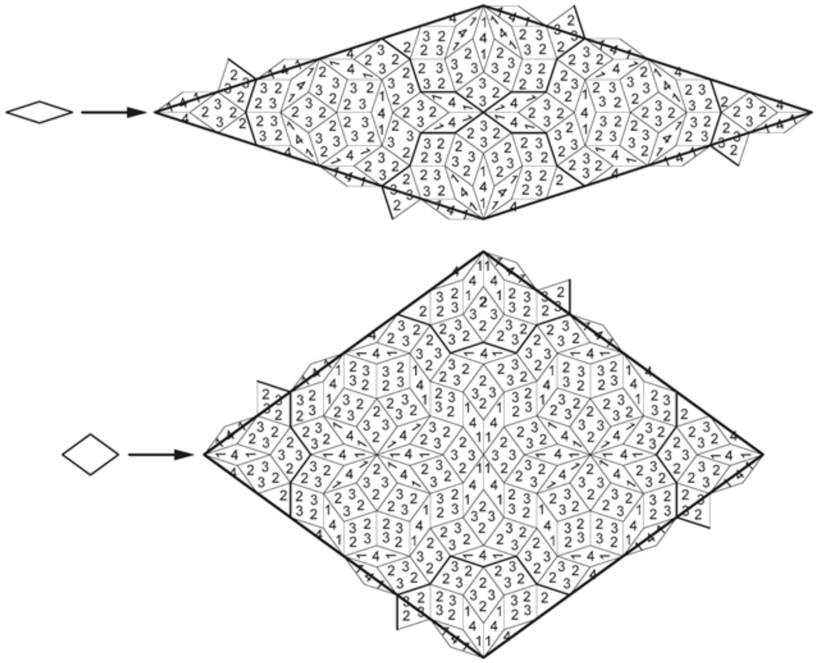

Fig. 4 Substitution rule for $n = 5$

odd n. Figure 5 shows the first image of rose R_2^1. Note, again, roses R_2^1 centered at all corners of the super-rhombuses.

3.3 Substitution Rules for $n = 7$

The substitution rule for $n = 7$ is shown in the Fig. 6. Similarly to case of $n = 5$, it is easy to verify that at each intersection of rhombuses, the sum of the angles is 2π.

3.4 Compositions of Super-Rhombus' Edges

Every Sub Rosa tiling is formed in such a way that the edge of the super-rhombus bisects all unit rhombuses along it. The length of this edge is $S(n)$, the scaling factor of the substitution.

Recall that we identify the edge substitution rule by the sequence $\Sigma(n)$ of the labels of the angles the edge bisects. In Table 1 is depicted the sequence $\Sigma(n)$ for small odd n, where symbol | denotes the middle point of the edge. The underlined values in the table represent rhombuses which are inside roses R_2^1 centered at the corners of the super-rhombus. There is a simple rule to form $\Sigma(n)$: The first half of $\Sigma(n)$ consists of the (underlined) sequence $1, 3, 5, \ldots, (n-2)$, followed by the mirror images of the underlined parts of $\Sigma(3), \Sigma(5), \ldots, \Sigma(n-2)$, that is, by 1, 31, 531, etc. The

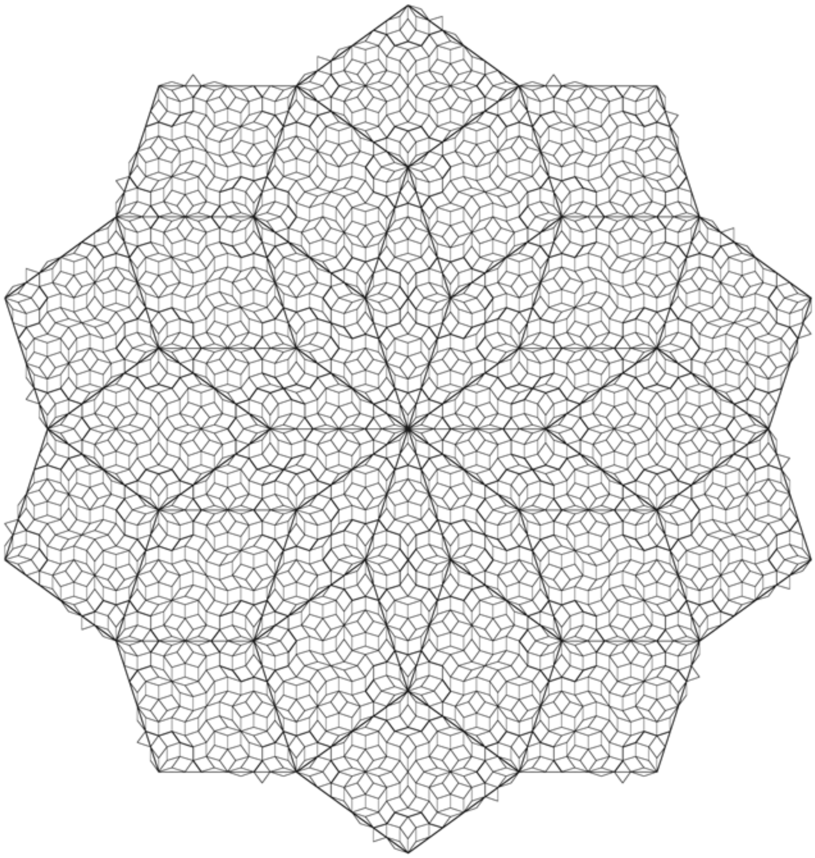

Fig. 5 Rose R_2^1 for $n = 5$ after one substitution

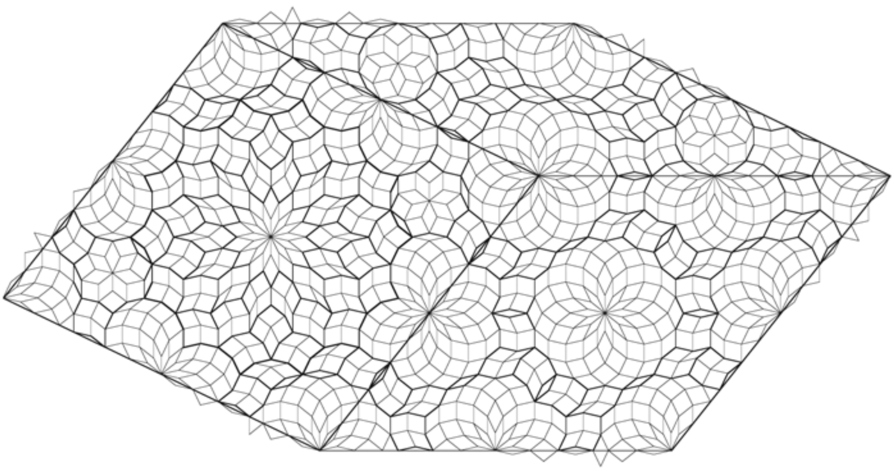

Fig. 6 Super-rhombuses of substitution rule for $n = 7$

Table 1 Composition of super-rhombus' edges for first n given in unit rhombuses

$\Sigma(3)$	1 | 1
$\Sigma(5)$	1-3-1 | 1-3-1
$\Sigma(7)$	1-3-5-1-3-1 | 1-3-1-5-3-1
$\Sigma(9)$	1-3-5-7-1-3-1-5-3-1 | 1-3-5-1-3-1-7-5-3-1
$\Sigma(11)$	1-3-5-7-9-1-3-1-5-3-1-7-5-3-1 | 1-3-5-7-1-3-5-1-3-1-9-7-5-3-1

second half of $\Sigma(n)$ is the mirror image of its first half. This blueprint provides the edge substitution rule for all odd n.

We choose all the rhombuses on the left side of the midpoint to be counted in and all rhombuses on the right side of midpoint to be counted out from the super-rhombus. Any other partition would work as well, as long as it is mirror symmetric.

Diagonal measure is the length of the diagonal of a unit rhombus $(k, n-k)$ that bisects k, and is denoted by $d_n(k)$. The diagonal measure is given by the formula

$$d_n(k) = 2\cos\left(k\frac{\pi}{2n}\right). \tag{2}$$

From $\Sigma(n)$ we can read the scaling factor $S(n)$ as a sum of diagonal measures $d_n(k)$. For example, from $\Sigma(7) = 1, 3, 5, 1, 3, 1, 1, 3, 1, 5, 3, 1$ we obtain that

$$S(7) = 2(d_7(1) + d_7(3) + d_7(5) + d_7(1) + d_7(3) + d_7(1)).$$

From the proposed general structure of $\Sigma(n)$ for all odd n, we get

$$S(n) = (n-1)d_n(1) + (n-3)d_n(3) + \cdots + 2d_n(n-2).$$

The scaling factor in (1) was inferred from this relation to diagonal measures (2).

4 Substitution Rules for Even *n*

In this section we consider briefly Sub Rosa tilings for even values of n. Now the scaling factor is different

$$S(n) = \frac{2}{1-\cos\left(\frac{\pi}{n}\right)}. \tag{3}$$

The smallest case $n = 2$ has scaling factor 2: it is the square substitution where the square is replaced by four squares. The resulting tiling is the regular square tiling, which is also the only edge-to-edge rhombic tiling in this case.

Consider an arbitrary even $n \geq 2$.

- *The edge substitution rule* The edges of enlarged rhombuses bisect some unit rhombuses and coincide with the edges of some unit rhombuses. In the latter case we say the edge bisects a zero rhombus, and indicate such situation by label 0 in

Table 2 Composition of super-rhombus' edges for first even n

$\Sigma(2)$	0|0
$\Sigma(4)$	0-2-0|0-2-0
$\Sigma(6)$	0-2-4-0-2-0|0-2-0-4-2-0
$\Sigma(8)$	0-2-4-6-0-2-0-4-2-0|0-2-4-0-2-0-6-4-2-0
$\Sigma(10)$	0-2-4-6-8-0-2-0-4-2-0-6-4-2-0|0-2 4-6-0-2-4-0-2-0-8-6-4-2-0
$\Sigma(12)$	0-2-4-6-8-10-0-2-0-4-2-0-6-4-2-0-8-6-4-2-0|0-2-4-6-8-0-2-4-6-0-2-4-0-2-0-10-8-6-4-2-0

$\Sigma(n)$. The sequence $\Sigma(n)$ is mirror symmetric and of even length. This means that, as in the odd case, the super-rhombuses match each other without gaps or overlaps if we count in the super-rhombus the bisected tiles in the first half of $\Sigma(n)$, and count out the bisected tiles in the second half of $\Sigma(n)$. The edge substitution rules for small even n are shown in Table 2.

– The starting pattern is again R_2^1. All super-rhombuses have a sector of R_2^1 at each vertex, but now the sector is aligned so that the first label of $\Sigma(n)$ is 0 rather than 1. This means that the image of R_2^1 contains at its center R_2^1 in its original orientation. In other words, no quarter rotation is needed to match consecutive generations, and the final tiling is a fixed point of the substitution.
– The general structure of $\Sigma(n)$ for even n is analogous to the odd case: The first half of $\Sigma(n)$ consists of the (underlined) sequence $0, 2, 4, \dots, (n-2)$, followed by the mirror images of the underlined parts of $\Sigma(2), \Sigma(4), \dots, \Sigma(n-2)$, that is, by 0, 20, 420, etc. The second half of $\Sigma(n)$ is the reversal of the first half. The obtained word is a palindrome of even length.

Rose R_2^1 for $n = 4$ after one substitution can be seen in Fig. 7. For $n = 6$ the substitution rule is shown in the Fig. 8.

5 General Case

We have shown substitution rules for small values of n. In the following we demonstrate that analogous substitutions exist for all values of n, thus proving Theorem 1. We use the explicit description of super-rhombuses' boundaries as in Tables 1 and 2. Then we use the method in [7,8] to prove that the interior can be tiled with the unit rhombuses.

We first set our notations. Let n be fixed. All angles will be expressed in units $\frac{\pi}{n}$. Number $2n$ is the full circle so that angles are considered modulo $2n$. Direction 0 is drawn horizontally to the right, so that directions $\frac{n}{2}$, n and $-\frac{n}{2}$ refer to up, left and down, respectively. For each direction x, the antiparallel direction $x+n$ is denoted by $\overleftarrow{x}$. Hence $\overleftarrow{\overleftarrow{x}} = x$.

In Tables 1 and 2 the structure of super-rhombuses' edges is given in terms of the unit rhombuses along edges. In the following it is more convenient to consider the sequence of unit vectors that enclose the interior that needs to be tiled. Each unit vector is represented by its direction, so the *boundary word* of the super-rhombus is the sequence of directions of the unit vectors on the boundary. This sequence is

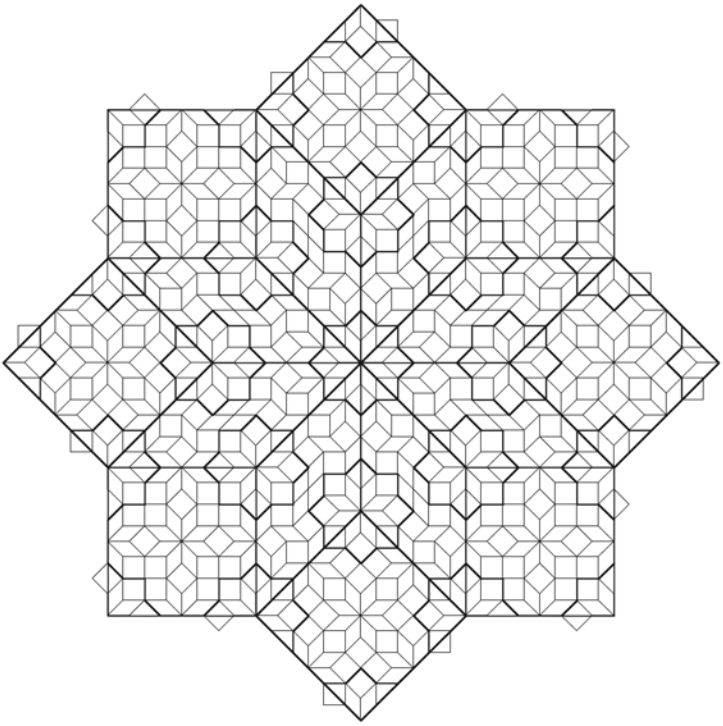

Fig. 7 Rose R_2^1 for $n = 4$ after one substitution

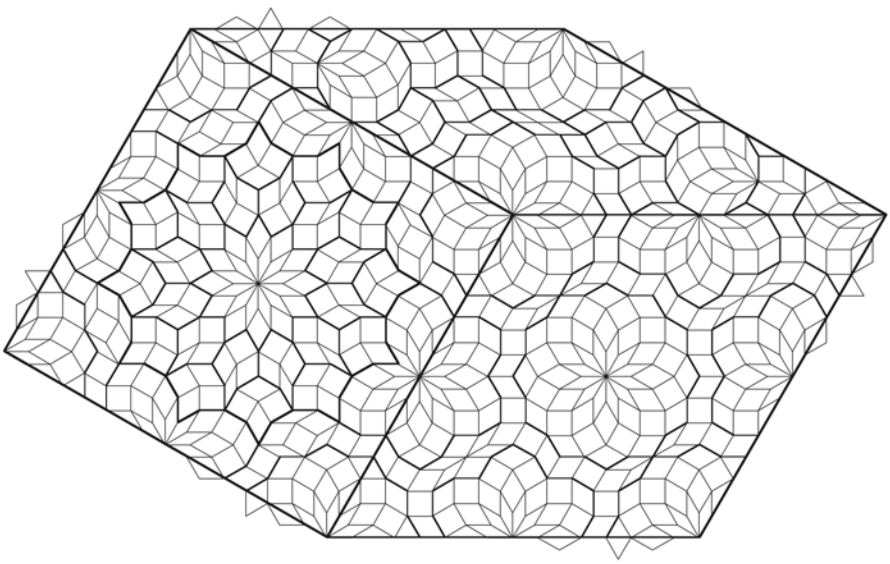

Fig. 8 Super-rhombuses of the substitution rule for $n = 6$

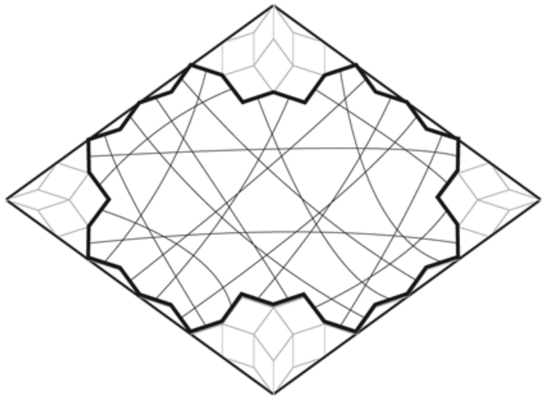

Fig. 9 The boundary of the region to be tiled with unit rhombuses, in the case $n = 5$ and super-rhombus $(2, 3)$. *Lines* connect the matching pairs of unit vectors along the boundary. Each crossing of the *lines* corresponds to a unit rhombus in the interior. In this case, each crossing provides a properly oriented rhombus which means that the crossing condition is satisfied

a cyclic word so that conjugate words uv and vu denote the same boundary, just with a different starting point. We read the boundary word counterclockwise. The unit rhombuses bisected by the super-edge are not included in interior to be tiled. This means that a unit rhombus $(a, n - a)$ on a super-edge of direction k contributes in the boundary word two unit vectors in directions $k + \frac{a}{2}$ and $k - \frac{a}{2}$, in this order.

In fact, we want to show that the interior can be tiled in such a way that sectors of rose R_2^1 appear centered at all four corners of the super-rhombus. The underlined symbols in Tables 1 and 2 are inside these roses so they get replaced by the boundary of the rose sector. As an example, Fig. 9 shows the boundary of the region to be tiled in the case of the $(2, 3)$ super-rhombus. Note how the boundary traces the sectors of the R_2^1 roses at the corners.

We use the standard notations on words. If u and v are two words then uv is their concatenation. A concatenation of k copies of word u is denoted as u^k. Though individual letters a are directions, and hence numbers modulo n, the power notation a^k represents the word $aa \dots a$ of length k (rather than number a to power k). We denote by u^R the *reversal* of word u, that is, the word obtained by writing the letters of u in the reverse order. Word u is a *palindrome* if $u^R = u$. The empty word is denoted as ε. It has length zero. Sometimes, for clarity, we write words with commas separating the letters. So $aabab$ and a, a, b, a, b denote the same word of length five.

For any direction x, we define the operation σ_x on words that increments each letter by constant x. This corresponds to turning the entire path by angle x. In particular, σ_n orients the path in the opposite direction. We extend the notation of antiparallelism to words so that $\overleftarrow{u} = \sigma_n(u)$ denotes the path antiparallel to u, that is, half turned u.

Since the same palindromic edge substitution is used on all four edges of a super-rhombus, it easily follows that the contributions on the boundary word by opposite super-edges are antiparallel to each other, and the same holds for the contributions by the rose sectors at opposite corners of the super-rhombus. Hence the boundary word is of the form $u\overleftarrow{u}$ where u is the word through half of the boundary. It follows from this symmetry that, in each direction a, the boundary contains equally many unit vectors in directions a and $\overleftarrow{a}$. This is the *balance condition* in [7].

It is also easy to see that the boundary does not cross itself. Only in the case of the $(1, n-1)$ super-rhombus for odd n the roses at the opposite $n-1$ corners touch in the middle, but without crossing each other. In the terminology of [7] the boundary is *simple*.

Based on [7], to prove that the interior can be tiled with the unit rhombuses it only remains to show that the boundary vectors can be *matched* in an appropriate way. Each letter a on the boundary word must be matched with an occurrence of the antiparallel letter $\overleftarrow{a}$. A *crossing* is formed by two matched pairs $a \frown \overleftarrow{a}$ and $b \frown \overleftarrow{b}$ if they occur in the circular boundary word in the interleaved order $\dots a \dots b \dots \overleftarrow{a} \dots \overleftarrow{b} \dots$. For each such crossing, it is required that the path $ab\overleftarrow{a}\overleftarrow{b}$ forms a rhombus in the counterclockwise direction, that is, $a < b < a + n \pmod{2n}$. This is the *crossing condition*.

The convex crossing condition used in [7] is slightly weaker as it also allows crossings between matched pairs $a \frown \overleftarrow{a}$ and $a \frown \overleftarrow{a}$ with the same labels. We prefer to forbid this because we then get unique matchings: In our setup, all occurrences of a and $\overleftarrow{a}$ are in the opposite halves of the boundary word. This implies that there is a unique way of matching the occurrences of a and $\overleftarrow{a}$ with each other without crossings. Indeed, the i'th occurrence of a must be paired with the i'th last occurrence of $\overleftarrow{a}$.

What needs to be checked is that this unique matching respects the crossing condition for all distinct directions a and b. This in mind, for any a and b we define the *projection* function $\pi_{a,b}$ that erases from a boundary word all letters except a, b, $\overleftarrow{a}$ and $\overleftarrow{b}$. The crossing condition of the boundary word u can be expressed equivalently as the requirement that all projections $\pi_{a,b}(u)$ satisfy the crossing condition. This turns out to be equivalent in our setup to the property that $\pi_{a,b}(u)$ defines a non-crossing cycle in the counterclockwise direction.

Example 1 Consider the boundary word of the (2, 3) rhombus in Fig. 9. Keeping the orientation of the figure, the directions of the unit vectors are from $\mathbb{Z} + \frac{1}{2}$.

Reading counterclockwise, starting where the leftmost rose segment ends, the boundary word is

$$\tfrac{-1}{2}, \tfrac{-3}{2}, \tfrac{-1}{2}, \tfrac{-3}{2} \,\Big|\, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{-3}{2}, \tfrac{-1}{2} \,\Big|\, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2} \,\Big|\, \tfrac{5}{2}, \tfrac{7}{2}, \tfrac{3}{2}, \tfrac{5}{2} \,\Big|$$
$$\tfrac{9}{2}, \tfrac{7}{2}, \tfrac{9}{2}, \tfrac{7}{2} \,\Big|\, \tfrac{-9}{2}, \tfrac{-7}{2}, \tfrac{9}{2}, \tfrac{-9}{2}, \tfrac{7}{2}, \tfrac{9}{2} \,\Big|\, \tfrac{-7}{2}, \tfrac{-9}{2}, \tfrac{-7}{2}, \tfrac{-9}{2} \,\Big|\, \tfrac{-5}{2}, \tfrac{-3}{2}, \tfrac{-7}{2}, \tfrac{-5}{2} \,\Big|$$

Vertical lines indicate the changes between the four rose segments and the four edge segments of the boundary. There are five different directions (when antiparallel directions are not counted separately), so there are $\binom{5}{2} = 10$ different pairs of directions that define projections $\pi_{a,b}$. For example, for $a = \frac{3}{2}$ and $b = \frac{5}{2}$, keeping in mind that $\overleftarrow{a} = \frac{-7}{2}$ and $\overleftarrow{b} = \frac{-5}{2}$, the projection by $\pi_{a,b}$ is

$$a\; a\; a\; b\; a\; b\; \overleftarrow{a}\; \overleftarrow{a}\; \overleftarrow{a}\; \overleftarrow{b}\; \overleftarrow{a}\; \overleftarrow{b}.$$

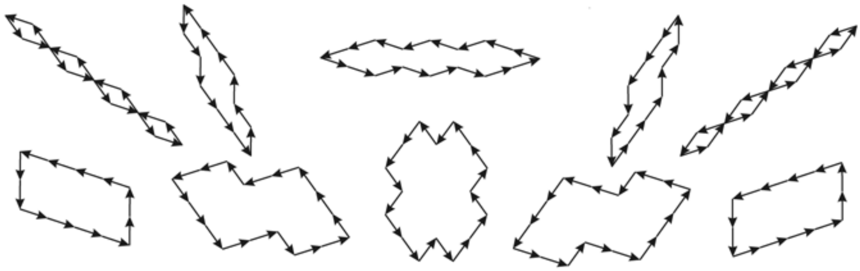

Fig. 10 All projections in pairs of distinct directions of the boundary of the (2, 3) super-rhombus. All paths are nonintersecting and counterclockwise, so the crossing condition is satisfied

This word defines a non-crossing boundary in the counterclockwise direction. Analogously we can determine the projections for all pairs a and b. The corresponding paths are shown in Fig. 10. The crossing condition is satisfied. □

The results in [7,8] guarantee that a region surrounded by a simple boundary that satisfies the balance condition and whose (unique) matching satisfies the crossing condition has a tiling by parallelograms ([7, Thm. 2]). In our setup the only non-trivial aspect to check is the crossing condition. In the following section we develop a convenient rewrite system to check this condition.

Remark 1 While [7] guarantees a tiling by parallelograms, it is easy to see that the proof in [7] provides a tiling by unit rhombuses if the boundary consists of unit length segments.

5.1 Rewrite System to Check the Crossing Condition

Our standing assumption in the following is that a and b are directions such that $a < b < a + n \pmod{2n}$ so that $ab\overleftarrow{a}\overleftarrow{b}$ is a proper rhombus in the counterclockwise direction. Let u, v be words over the alphabet $\{a, b, \overleftarrow{a}, \overleftarrow{b}\}$. Consider the following eight *rewrite* rules:

$$\begin{array}{llll} ab \longrightarrow ba, & b\overleftarrow{a} \longrightarrow \overleftarrow{a}b, & \overleftarrow{a}\overleftarrow{b} \longrightarrow \overleftarrow{b}\overleftarrow{a}, & \overleftarrow{b}a \longrightarrow a\overleftarrow{b}, \\ a\overleftarrow{a} \longrightarrow \varepsilon, & \overleftarrow{a}a \longrightarrow \varepsilon, & b\overleftarrow{b} \longrightarrow \varepsilon, & \overleftarrow{b}b \longrightarrow \varepsilon, \end{array} \tag{4}$$

An application of rule $x \longrightarrow y$ on word u means that we replace an occurrence of subword x in u by y. More precisely, if $u = u_1 x u_2$ and $v = v_1 y v_2$ and $x \longrightarrow y$ is a rewrite rule then u derives v and we write $u \rightarrow v$. We denote the transitive, reflexive closure of $\rightarrow$ by $\rightarrow^*$, so that $u \rightarrow^* v$ means that u can be turned into v by a sequence of rewrites. Iterating rule $ab \longrightarrow ba$, for example, allows us to move b's any number of positions to the left on a word containing only a's and b's, or to move a's to the right on such a word. In other words, $au \rightarrow^* ua$ and $ub \rightarrow^* bu$ for any word u that only contains letters a and b.

The top four rewrite rules in (4) correspond to snapping two consecutive edges of the rhombus $ab\overleftarrow{a}\overleftarrow{b}$ in reverse order. The last four rules allow eliminating a letter and its reversal that are next to each other.

Suppose $u \to v$ using the first rewrite rule $ab \longrightarrow ba$. If v satisfies the crossing condition, so does u with the same matchings. Indeed, the only new crossing in u, not present in v, connects pairs $a \frown \overleftarrow{a}$ and $b \frown \overleftarrow{b}$ of symbols that appear in u in the (circular) order $\ldots ab \ldots \overleftarrow{a} \ldots \overleftarrow{b} \ldots$. Such order is allowed by the crossing condition. The same argument applies to all four rewrite rules that reverse the order of two symbols.

Suppose then that $u \to v$ using the eliminating rewrite rule $a\overleftarrow{a} \longrightarrow \varepsilon$. If v satisfies the crossing condition, so does u when we connect the two letters in the eliminated pair $a\overleftarrow{a}$ with each other, and match all other letters in the same way as they were matched in v. In this way u has exactly the same crossings as v. The same argument applies to any other eliminating rewrite rule.

We have seen that if $u \to^* v$ and if v satisfies the crossing condition then also u satisfies the crossing condition. In particular, $u \to^* \varepsilon$ guarantees that u satisfies the crossing condition. In fact, it is not difficult to see that this condition is also necessary:

Lemma 1 *Let u be a word over the alphabet $\{a, b, \overleftarrow{a}, \overleftarrow{b}\}$. Then u has a matching that satisfies the crossing condition if and only if $u \to^* \varepsilon$.*

Proof We have seen above that $u \to^* \varepsilon$ implies that there is matching in u that satisfies the crossing condition. Let us prove the converse direction. Define a partial order among words with matchings: $u < v$ if u is shorter than v, or if u and v have the same lengths but u has fewer crossings than v.

Let u be a non-empty word with a matching that satisfies the crossing condition. We want to prove that $u \to v$ for some $v < u$ that also satisfies the matching condition.

Assume first that u contains two consecutive letters that are matched with each other. Hence the letters are antiparallel to each other, and can be eliminated by a rewrite rule. This produces a shorter word v, so $v < u$. The remaining letters in v can be matched exactly as in u, satisfying the crossing condition.

Assume next that u contains two consecutive letters xy whose connections cross each other. By the crossing condition, $xy\overleftarrow{x}\overleftarrow{y}$ must be a proper rhombus in the counter-clockwise direction, so $xy \longrightarrow yx$ is a valid rewrite rule. When we apply it to u, and keep letter matchings unchanged, we obtain v that has the same length but one less crossing than u. Hence $u \to v$ and $v < u$. Clearly v satisfies the crossing condition since we obtained it from u by only untangling one crossing.

Finally, assume that u has no consecutive letters that are connected to each other or whose connections would cross each other. Let x and y be two letters in u that are either connected to each other or whose connections cross, and assume their (cyclic) distance d is as short as possible. The letters are not next to each other so there is a letter z between them. But z cannot be connected to another letter $\overleftarrow{z}$ between x and y because then the distance from z to $\overleftarrow{z}$ would be shorter than d, contradicting the choice of x and y. Hence the connection of z necessarily crosses the x, y pair, and hence it crosses either the connection of x or the connection of y. But the distance from z to both x and y is shorter than d, which again contradicts the minimality of d.

We have proved the claim that $u \rightarrow v$ for some $v < u$ that also satisfies the matching condition. Now, if $v \neq \varepsilon$ the argument can inductively applied on v. By iterating the argument we obtain a sequence $u \rightarrow v \rightarrow \dots$. The partial order $<$ does not admit infinite decreasing chains so ε must be eventually reached. □

Remark 2 In our setup the boundary u of the tileable region has all the symmetries of the enlarged rhombus: dihedral group D_4 or D_2 if the rhombus is or is not a square, respectively. The unique matching also respects these symmetries, and so do all projections $\pi_{a,b}(u)$. Shrinking the remaining tileable region by adding a unit rhombus tile with two edges on the boundary corresponds to the application of a rewrite rule on some $\pi_{a,b}(u)$. The analogous rewrite can be done on all symmetric positions, thus reducing the tileable region while keeping its symmetries. By iterating this process we obtain a tiling of the interior of the enlarged rhombus that has all the symmetries of the initial rhombus.

A detail to observe in this process is that each symmetry of the boundary takes any pair of consecutive edges to a disjoint pair, or to the identical pair of consecutive edges, but never to a pair that shares exactly one edge with the original pair. For this reason all symmetric positions can be rewritten independently of each other. In contrast, the reader may consider, for example, tiling the regular octagon of unit sides: the reflection symmetries prevent a fully symmetric tiling of the interior by unit rhombuses.

Notice that $u \rightarrow^* v$ implies that $\overleftarrow{u} \rightarrow^* \overleftarrow{v}$. This is because for each rewrite rule $x \longrightarrow y$ there is also the rewrite rule $\overleftarrow{x} \longrightarrow \overleftarrow{y}$ in our toolbox (4). This also implies that if $u \rightarrow^* v$ and $u \rightarrow^* v^R$ for some word v then $u\overleftarrow{u}$ satisfies the crossing condition. Indeed, $u\overleftarrow{u} \rightarrow^* v^R\overleftarrow{v} \rightarrow^* \varepsilon$, where the last steps use elimination rewrites. In particular, if a palindrome can be derived from u then $u\overleftarrow{u}$ satisfies the crossing condition.

Example 2 Let $u = aaabab$ so that $u\overleftarrow{u}$ is the boundary word from Example 1. Since $aaabab \rightarrow^* baaaab$ we have that $u\overleftarrow{u} \rightarrow^* baaaab\,\overleftarrow{b}\overleftarrow{a}\overleftarrow{a}\overleftarrow{a}\overleftarrow{a}\overleftarrow{b} \rightarrow^* \varepsilon$. Analogously, all ten projections shown in Fig. 10 can be reduced into words of the form $p\overleftarrow{p}$ for some palindromes p, and hence into the empty word ε. □

We are interested in the matching condition on boundary words that are of the type $u\overleftarrow{u}$, by performing rewrites on the half word u. As we work with the half boundary, there is yet another useful operation on the words: If $u = xy$ is a concatenation of two words then we may swap the order of x and y while changing x into $\overleftarrow{x}$. We write $xy \rightsquigarrow y\overleftarrow{x}$. When such operation is performed on both halves of $u\overleftarrow{u}$, the (circular) word remains unchanged, that is, if $u \rightsquigarrow v$ then $u\overleftarrow{u}$ and $v\overleftarrow{v}$ are the same (circular) words. Indeed, $u\overleftarrow{u} = xy\overleftarrow{x}\overleftarrow{y}$ and $v\overleftarrow{v} = y\overleftarrow{x}\overleftarrow{y}x$ for $v = y\overleftarrow{x}$.

We denote $u \Rightarrow v$ if $u \rightarrow v$ or $u \rightsquigarrow v$, and by $\Rightarrow^*$ we denote the reflexive transitive closure of $\Rightarrow$.

Lemma 2 *If $u \Rightarrow^* v \rightarrow^* v^R$ for some v then $u\overleftarrow{u}$ satisfies the crossing condition.*

Proof As $v\overleftarrow{v} \rightarrow^* v^R\overleftarrow{v} \rightarrow^* \varepsilon$, by Lemma 1 we know that $v\overleftarrow{v}$ satisfies the crossing condition. For any $x \Rightarrow y$, if $y\overleftarrow{y}$ satisfies the crossing condition so does $x\overleftarrow{x}$. Namely, if $x \to y$ then $x\overleftarrow{x} \to y\overleftarrow{y}$, and if $x \rightsquigarrow y$ then $x\overleftarrow{x}$ and $y\overleftarrow{y}$ are the same circular word. Hence, $u \Rightarrow^* v$ implies that $u\overleftarrow{u}$ satisfies the crossing condition because $v\overleftarrow{v}$ does. □

All our proofs of tileability use Lemma 2. We start with a half u of the boundary word $u\overleftarrow{u}$, and derive from u a new word v using operations $\to$ and $\rightsquigarrow$. Then we operate on v using only rewrite operations $\to$ to reach its reversal v^R. Of course we could always work on the full boundary $u\overleftarrow{u}$ but working on u reduces the size of expressions and the number of operations by half. We frequently use

$$xyx^R \Rightarrow^* yx^R\overleftarrow{x} \Rightarrow^* y,$$

and simply cancel a prefix and a suffix from u if they are reversals of each other.

The following particular case will be used multiple times, so we state it as a separate lemma.

Lemma 3 *Assume our usual hypothesis that $ab\overleftarrow{a}\overleftarrow{b}$ is a rhombus in the counterclockwise orientation. If $u = (ba)^i(b\overleftarrow{a})^j$ where $j \geq 1$ then $u \Rightarrow^* v \to^* v^R$ for some v. In particular, $u\overleftarrow{u}$ satisfies the crossing condition.*

Proof We apply the rewrite rules in (4) that move a to the right and $\overleftarrow{a}$ to the left over b's, and cancel a and $\overleftarrow{a}$ as they meet.

If $i < j$ then

$$(ba)^i\ (b\overleftarrow{a})^j \Rightarrow^* b^ia^i\ \overleftarrow{a}^i b^i\ (b\overleftarrow{a})^{j-i} \Rightarrow^* b^{2i}\ (b\overleftarrow{a})^{j-i} \to^* (\overleftarrow{a}b)^{j-i}\ b^{2i},$$

and the last two words are reverses of each other. In the last steps we moved $2i+1$ letters b from the beginning of the word to the end. If $i \geq j \geq 1$,

$$(ba)^i(b\overleftarrow{a})^j \Rightarrow^* (ba)^{i-j}\ b^ja^j\ \overleftarrow{a}^j b^j \Rightarrow^* (ba)^{i-j}b^{2j} \to^* b^{2j}(ab)^{i-j},$$

where the last steps of the derivation move $2j-1$ copies of b from the end of the word to the beginning. Also here the last words are reverses of each other. Lemma 2 confirms that $u\overleftarrow{u}$ satisfies the crossing condition. □

5.2 Boundary Words of Super-Rhombuses

Let us fix $k \in \{1, 2, \ldots, n-1\}$ and consider the super-rhombus S of type $(k, n-k)$. Let us orient S on the plane with its bottom edge horizontal and corner k at the left end of the base. Super-rhombuses $(k, n-k)$ and $(n-k, k)$ are identical, so it is sufficient to consider only one of them for each k. We name the rose and edge segments of the boundary of S by letters A through H, as shown in Fig. 11. Segments A, C, E and G are *edge segments* and B, D, F and H are *rose segments*. In the counterclockwise orientation, the directions of edges A and C are 0 and k, respectively.

Recall that, for odd n, the edge segments bisect the sequence

$$\tau(n) = 1\,|\,31\,|\,531\,|\,\ldots|\,(n-4)(n-6)\ldots31\,\|\,13\ldots(n-6)(n-4)\,|\,\ldots\\ |\,135\,|\,13\,|\,1$$

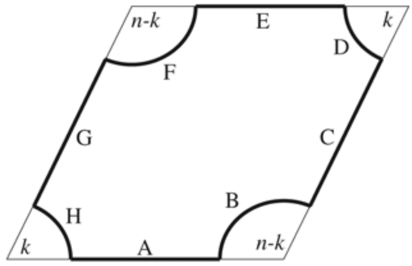

Fig. 11 The eight segments of the boundary

of unit rhombuses, where shape $(m, n - m)$ is represented simply as m, and the vertical lines are added to emphasize the structure of the sequence. When n is even, the sequence of bisected unit rhombuses is

$$\tau(n) = 0\,|\,20\,|\,420\,|\,\ldots|\,(n-4)(n-6)\ldots20\,||\,02\ldots(n-6)(n-4)\,|\\ \ldots|\,024\,|\,02\,|\,0$$

where 0 denotes a single unit edge along the edge segment. The sequences are the non-underlined parts of $\Sigma(n)$, as shown in Tables 1 and 2. Remark that the sequences are palindromes, and that all numbers m in the sequences have the same parity as n. Let

$$f(m) = \frac{n-m}{2} - 1,$$

so that m appears $f(m)$ times in both halves of $\tau(n)$.

Observe the following structure of $\tau(n)$: let $k, m \in \{1, 3, \ldots, n-2\}$ or $k, m \in \{0, 2, \ldots, n-2\}$ when n is odd or even, respectively, and $k < m$. Then the projected sequence obtained by erasing all other numbers from $\tau(n)$ has the form

$$k^i(mk)^j(km)^jk^i, \text{ where } i = f(k) - f(m) \geq 1 \text{ and } j = f(m) \geq 0. \tag{5}$$

On the horizontal segment A, each occurrence of m in $\tau(n)$ contributes symbols $\frac{m}{2}, \frac{-m}{2}$ on the boundary word, when $m \neq 0$. This is because A is in direction 0, so that each m represents a unit rhombus $(m, n-m)$ whose edges are in directions $\frac{m}{2}$ and $\frac{-m}{2}$. Each occurrence of $m = 0$ contributes a single 0 on the boundary word. When n is even, all symbols on the boundary word are integers; when n is odd they are from the set $\mathbb{Z} + \frac{1}{2}$. This is easily seen to be the case on all eight segments in Fig. 11, including both the edge and the rose segments. We have that in the case of odd n the directions of the unit vectors present on the boundary are not parallel to any edge of S. In contrast, on even n, each occurrence of 0 in $\tau(n)$ yields a unit vector parallel to an edge. In both even and odd cases, directions $\frac{\pm n}{2}$ perpendicular to A are possible.

For any direction x, where $x \in \mathbb{Z}$ or $x \in \mathbb{Z} + \frac{1}{2}$ if n is even or odd, respectively, we introduce the notation $\diamond_x = m$ if the unit rhombus $(m, n-m)$ on segment A contributes direction x on the boundary word. More precisely, $\diamond_x = m$ for all directions x such that $\pm 2x \equiv m \pmod{2n}$, and $0 \leq m \leq n$. We also define

$$\bar{x} = \begin{cases} x, & \text{if } \pm 2x \equiv \diamond_x \pmod{4n}, \\ \overleftarrow{x}, & \text{if } \pm 2x \equiv \diamond_x + 2n \pmod{4n}, \end{cases}$$

so that, among the two orientations x and $\overleftarrow{x}$, the actual contribution by rhombus $(\diamond_x, n - \diamond_x)$ on the boundary word along segment A is $\bar{x}$.

Let us discuss next briefly the rose segments. The four segments combined in the order H, F, D, B form the full rose R_2^1 in the clockwise orientation. Each unit vector direction appears twice in the rose. In R_2 the directions would appear perfectly ordered in the increasing order, but omitting the outermost ring in R_2^1 swaps consecutive pairs of directions. In segment B, directions $\frac{n}{2}$ and $k - \frac{n}{2}$ both appear once (as the second, and the second last direction, respectively), while the directions x in the interval $k - \frac{n}{2} < x < \frac{n}{2}$ appear twice in B.

Let $a, b \in \mathbb{Z}$ (even n) or $a, b \in \mathbb{Z} + \frac{1}{2}$ (odd n) be two distinct directions of unit vectors on the boundary word. We denote by $\alpha(a, b)$, $\beta(a, b)$, $\gamma(a, b)$ and $\delta(a, b)$ the $\pi_{a,b}$ projections of the boundary word segments on A, B, C and D, respectively. We denote their concatenation by u:

$$u = \alpha(a, b)\beta(a, b)\gamma(a, b)\delta(a, b).$$

The $\pi_{a,b}$ projection of the entire boundary is $u\overleftarrow{u}$.

It is easy to see that $\alpha(a, b)$ depends on the directions a and b as follows:

- If $2a \equiv n \pmod{2n}$ so that a is orthogonal to segment A, then direction a does not appear in $\alpha(a, b)$ and hence

$$\alpha(a, b) = \bar{b}^i,$$

  where $i = 2f(\diamond_b) = n - \diamond_b - 2$. Analogously, if $2b \equiv n \pmod{2n}$ then $\alpha(a, b)$ does not contain direction b.
- Otherwise $\diamond_a, \diamond_b < n$, so we can infer $\alpha(a, b)$ from (5). If $\diamond_a < \diamond_b$ then

$$\alpha(a, b) = \bar{a}^{f(\diamond_a) - f(\diamond_b)} (\bar{b}\bar{a})^{f(\diamond_b)} (\bar{a}\bar{b})^{f(\diamond_b)} \bar{a}^{f(\diamond_a) - f(\diamond_b)}.$$

  Notice that this word is a palindrome. Analogously, if $\diamond_b < \diamond_a$ then

$$\alpha(a, b) = \bar{b}^{f(\diamond_b) - f(\diamond_a)} (\bar{a}\bar{b})^{f(\diamond_a)} (\bar{b}\bar{a})^{f(\diamond_a)} \bar{b}^{f(\diamond_b) - f(\diamond_a)}.$$

  The last possibility is $\diamond_a = \diamond_b$. Now

$$\alpha(a, b) = (\bar{a}\bar{b})^{2f(\diamond_a)} \text{ or } \alpha(a, b) = (\bar{b}\bar{a})^{2f(\diamond_a)}.$$

The word $\gamma(a, b)$ along segment C is similar. Because the direction of C is k, directions a and b are oriented with respect to C in the same way as $a - k$ and $b - k$ are oriented with respect to A. We then have

$$\gamma(a, b) = \sigma_k(\alpha(a - k, b - k)).$$

5.3 Case Analysis

We are interested to analyze the word $u = \alpha(a, b)\beta(a, b)\gamma(a, b)\delta(a, b)$ which is the first half of the projection of the boundary word on directions a and b, and to show that $u\overleftarrow{u}$ satisfies the crossing condition. There are a number of cases to analyze depending on the relationship between directions a, b and k.

Note that we can reduce the number of cases due to symmetries. Because a and $\overleftarrow{a}$, and b and $\overleftarrow{b}$ define the same projections, we only need to consider one choice of each. We can also swap a and b if needed. We can therefore assume that

$$-\frac{n}{2} < a < b \leq \frac{n}{2}.$$

With this choice, $ab\overleftarrow{a}\overleftarrow{b}$ is a proper rhombus oriented counterclockwise, and the rewrite rules in (4) can be used.

Case 1: ***a*** **or** ***b*** **is perpendicular to a side of super-rhombus** ***S***

By suitably orienting S, and possibly replacing b by $\overleftarrow{b}$ or a by $\overleftarrow{a}$, we can assume that $b = \frac{n}{2}$ and $-\frac{n}{2} < a < \frac{n}{2}$. We can further assume that $k \leq \frac{n}{2}$. (If not, we flip the rhombus and consider $n - k$ instead of k.) In this case we have $\alpha(a, b) = a^i$ for some i.

(a) Assume $a > k - \frac{n}{2}$. Now $\beta(a, b) \in \{baa, aba\}$ and $\delta(a, b) = b$. In any case $\beta(a, b) \rightarrow^* baa$. Word $\gamma(a, b)$ is either some palindrome p containing only letters a and b (if $\diamond_{a-k} \neq \diamond_{b-k}$), or $\gamma(a, b) = (ba)^j$ for some j (if $\diamond_{a-k} = \diamond_{b-k}$). In the first case

$$u \Rightarrow^* a^i\ baa\ p\ b \Rightarrow^* b\ a^{i+2}\ p\ b \rightarrow^* b\ p\ a^{i+2}\ b.$$

The last two words are reversals of each other, so by Lemma 2 we know that the boundary word $u\overleftarrow{u}$ satisfies the crossing condition. In the second case,

$$u \Rightarrow^* a^i\ baa\ (ba)^j\ b \Rightarrow^* b\ a^{i+1}\ a(ba)^j\ b \rightarrow^* b\ a(ba)^j\ a^{i+1}\ b.$$

Again, the last two derived words are reversals of each other.

(b) Next we assume that $a = k - \frac{n}{2}$. Now $\beta(a, b) = ba$, $\gamma(a, b) = b^j$ for some j, and $\delta(a, b) = \overleftarrow{a}b$, so that

$$u = a^i\ ba\ b^j\ \overleftarrow{a}b \Rightarrow^* a^i\ b^{j+1}\ a\overleftarrow{a}\ b \Rightarrow a^i b^{j+2} \rightarrow^* b^{j+2}a^i.$$

The last two words are reversals of each other.

(c) If $-\frac{n}{2} < a < k - \frac{n}{2}$ then $\beta(a, b) = b$ and $\delta(a, b) \in \{\overleftarrow{a}\overleftarrow{a}b, \overleftarrow{a}b\overleftarrow{a}\}$. As $\diamond_{a-k} \neq \diamond_{b-k}$, we know that $\gamma(a, b)$ is some palindrome p that contains letters $\overleftarrow{a}$ and b only.

$$u \Rightarrow^* a^i\ b\ p\ \overleftarrow{a}\overleftarrow{a}b \Rightarrow^* b\ a^i\ \ p\ \overleftarrow{a}\overleftarrow{a}b \Rightarrow^* a^i\ \ p\ \overleftarrow{a}\overleftarrow{a} \Rightarrow^* p\ \overleftarrow{a}^{i+2} \rightarrow^* \overleftarrow{a}^{i+2}\ p.$$

Also here, the last two words are reversals of each other.
In the remaining cases we can now assume that a and b are not perpendicular to the sides of S.

Case 2: $\diamond_a = \diamond_b$

Let us now assume that directions a and b are from the same unit rhombus on some edge segment. In a suitable orientation of S this means that $a = -b$. By symmetries, we can also assume that $0 < b < \frac{n}{2}$ and $k \leq \frac{n}{2}$. Now $\alpha(a, b) = (ba)^i$ for $i = 2f(\diamond_b) = n - 2b - 2$.

(a) Assume $a > k - \frac{n}{2}$. We have $\beta(a, b) \in \{bbaa, baba\}$ and $\delta(a, b) = \varepsilon$, the empty word. Because $\diamond_{a-k} = 2|a - k| = 2k + 2b$ and $\diamond_{b-k} = 2|b - k|$, it follows that $\diamond_{a-k} - \diamond_{b-k}$ is $4b$ or $4k$, depending on whether $b < k$ or $b > k$, respectively. In any case, $\diamond_{a-k} > \diamond_{b-k}$, so we have from (5) that $\gamma(a, b) = b^s(ab)^t(ba)^t b^s$ for some s, t. In fact, $s = \frac{1}{2}(\diamond_{a-k} - \diamond_{b-k}) = \min\{2b, 2k\}$.

If $s \geq 2$ then

$$\begin{aligned} u &\Rightarrow^* (ba)^i\ bbaa\ b^s(ab)^t(ba)^t b^s \\ &\Rightarrow^* b^s(ab)^{i+2}\ (ab)^t(ba)^t b^s \\ &\Rightarrow^* (ab)^{i+2} \\ &\rightarrow^* (ba)^{i+2}. \end{aligned}$$

In the third step we eliminated $b^s(ab)^t$ and its reversal from the prefix and the suffix of the word, respectively. The last words are reversals of each other so Lemma 2 applies.

Suppose then $s < 2$. Because $s = \min\{2b, 2k\}$, we see that $s < 2$ happens if and only if $b = \frac{1}{2}, a = -\frac{1}{2}$. But in this case, the rose segment B contains a unit vector in direction a between two unit vectors in direction b, so that $\beta(a, b) = baba$. (This is, interestingly, the reason why the rose segments come from the rose $R_2^{\frac{1}{2}}$ instead of the full rose R_2: in the full rose R_2 we would have $bbaa$ instead of $baba$, which would make the crossing condition fail in the case $b = \frac{1}{2}, a = -\frac{1}{2}$.) Now we have $s = 1$ and

$$\begin{aligned} u &= (ba)^i\ baba\ b(ab)^t(ba)^t b \\ &= (ba)^{i+t+2}\ bb\ (ab)^t \\ &\Rightarrow^* (ba)^{i+2}\ bb \\ &\rightarrow^* bb\ (ab)^{i+2}, \end{aligned}$$

where the last two derived words are reverses of each other.

(b) The other possibility is that $-\frac{n}{2} < a < k - \frac{n}{2}$. (Note that the case $a = k - \frac{n}{2}$ is covered by Case 1 because then a is perpendicular to segment C of S.) Now $\beta(a, b) = bb$ and $\delta(a, b) = \overleftarrow{a}\overleftarrow{a}$. Also we can calculate $\diamond_{a-k} = 2(a - k + n) = 2(n - k - b)$ and $\diamond_{b-k} = 2|b - k|$ so that $\diamond_{a-k} - \diamond_{b-k} = 2\min\{n - 2b, n - 2k\} \geq 0$.

Assume first that $k = \frac{n}{2}$ so that $\diamond_{a-k} = \diamond_{b-k}$. We have $\gamma(a, b) = (\overleftarrow{a}b)^j$ for some j, so

$$\begin{aligned} u &= (ba)^i\ bb\ (\overleftarrow{a}b)^j\ \overleftarrow{a}\overleftarrow{a} \\ &\Rightarrow^* (ba)^i\ (b\overleftarrow{a})^{j+2} \end{aligned}$$

The obtained word is of the form covered by Lemma 3.

Assume then that $k < \frac{n}{2}$ so that $\diamond_{a-k} > \diamond_{b-k}$. Then $\gamma(a, b) = b^s(\overleftarrow{a}b)^t(b\overleftarrow{a})^t b^s$ for some s, t. We derive

$$\begin{aligned} u &= (ba)^i\ bb\ b^s(\overleftarrow{a}b)^t\ (b\overleftarrow{a})^t b^s\ \overleftarrow{a}\overleftarrow{a} \\ &\Rightarrow^* b^s\ (ba)^i\ bb\ (\overleftarrow{a}b)^t\ (b\overleftarrow{a})^t\ \overleftarrow{a}\overleftarrow{a}\ b^s \\ &\Rightarrow^* (ba)^i\ (b\overleftarrow{a})^{2t+2}. \end{aligned}$$

Lemma 3 applies to the derived word.

From now on we can assume that neither Case 1 nor Case 2 applies.

Case 3: Directions a and b appear in the same rose segment B

Now $k - \frac{n}{2} < a < b < \frac{n}{2}$ and $0 < k < n$. (Note that we allow $k > \frac{n}{2}$ so that we do not assume the angle between segments A and B to be obtuse.) We can also assume that $b > 0$ because otherwise we can reflect the super-rhombus to swap segments A and C.

In this setup we have $\beta(a, b) \in \{bbaa, baba\}$ and $\delta(a, b) = \varepsilon$. Moreover, $\alpha(a, b)$ and $\gamma(a, b)$ are palindromes containing only letters a and b, and $\diamond_a = 2|a|$, $\diamond_b = 2|b| = 2b$, $\diamond_{a-k} = 2|a - k|$ and $\diamond_{b-k} = 2|b - k|$. Because Case 2 does not apply, we have $\diamond_a \neq \diamond_b$ and $\diamond_{a-k} \neq \diamond_{b-k}$.

(a) Assume first that $\diamond_a > \diamond_b$. Then $|a| > b > 0$ but $a < b$, which clearly implies that $a < 0$. Then also

$$\diamond_{a-k} = 2|a - k| = 2(|a| + k) > 2(b + k) \geq 2|b - k| = \diamond_{b-k}.$$

Now $\alpha(a, b) = b^i(ab)^j(ba)^j b^i$, and $\gamma(a, b) = b^s(ab)^t(ba)^t b^s$ for some $i, s \geq 1$ and $j, t \geq 0$. More precisely,

$$\begin{aligned} s &= f(\diamond_{b-k}) - f(\diamond_{a-k}) = |a - k| - |b - k| = k - a - |b - k|, \\ i &= f(\diamond_b) - f(\diamond_a) = |a| - |b| = -a - b. \end{aligned}$$

In particular, $s - i = k + b - |b - k| > 0$ so that $s > i \geq 1$. Also, because $\diamond_{a-k} = -2(a - k) > -2a = \diamond_a$, we have that $t < j$. We derive

$$\begin{aligned} u &\Rightarrow^* b^i(ab)^j(ba)^j b^i\ bbaa\ b^s(ab)^t(ba)^t b^s \\ &\Rightarrow^* (ab)^j(ba)^j b^{i+2}\ abab\ b^{s-2}(ab)^t(ba)^t b^{s-i} \\ &\Rightarrow^* (ab)^j(ba)^j b^{2s}\ abab\ (ab)^t(ba)^t \\ &\Rightarrow^* (ab)^{j-t}(ba)^j b^{2s}\ (ab)^{t+2} \\ &\Rightarrow^* (ba)^{2j-t} b^{2s}\ (ab)^{t+2} \end{aligned}$$

$$\begin{aligned}&\Rightarrow^* (ba)^{2j-2t-2}b^{2s}\\&\rightarrow^* b^{2s}(ab)^{2j-2t-2}.\end{aligned}$$

The last two words are reversals of each other.

(b) Assume next that $\diamond_a < \diamond_b$ and $\diamond_{a-k} < \diamond_{b-k}$. Then $k-b < k-a \leq |k-a| < |k-b|$ so that $k-b < 0$, that is, $b > k$.

Now $\alpha(a,b) = a^i(ba)^j(ab)^ja^i$, for $i = f(\diamond_a) - f(\diamond_b)$ and $j = f(\diamond_b)$, and $\gamma(a,b) = a^s(ba)^t(ab)^ta^s$, for $s = f(\diamond_{a-k}) - f(\diamond_{b-k})$ and $t = f(\diamond_{b-k})$. We have

$$\begin{aligned}t-j &= |b|-|b-k| = b-(b-k) = k > 0, \text{ and}\\ i-s &= |b|-|a|-(|b-k|-|a-k|)\\ &= b-(b-k)+|a-k|-|a| \geq k+(|a|-k)-|a| = 0,\end{aligned}$$

so that $t > j$ and $i \geq s \geq 1$.

Assume first that $i \geq 2$. Then

$$\begin{aligned}u &\Rightarrow^* a^i(ba)^j(ab)^ja^i\ bbaa\ a^s(ba)^t(ab)^ta^s\\ &\Rightarrow^* a^{i-s}(ba)^j(ab)^ja^i\ bbaa\ a^s(ba)^t(ab)^t\\ &\Rightarrow^* (ba)^j(ab)^j\ a^{2i-s}\ bbaa\ a^s(ba)^t(ab)^t\\ &\Rightarrow^* (ab)^{j+2}\ a^{2i}(ba)^t(ab)^{t-j}\\ &\Rightarrow^* (ab)^{j+2}\ a^{2i}\ (ba)^{2t-j}\\ &\Rightarrow^* a^{2i}\ (ba)^{2t-2j-2}\\ &\rightarrow^* (ab)^{2t-2j-2}a^{2i}.\end{aligned}$$

Note that we used the fact that $2i-s \geq 2$ on the fourth derivation line to change $aabb$ into $abab$. The last two words are reversals of each other.

Consider then the case $i = 1$, so that also $s = 1$. But $i = b-|a| = 1$ and $s = b-k-|a-k| = 1$ happen simultaneously only if $b = a+1$. In this case, $\beta(a,b) = baba$ as the path follows rose R_2^1 instead of rose R_2. We derive

$$\begin{aligned}u &= a(ba)^j(ab)^ja\ baba\ a(ba)^t(ab)^ta\\ &\Rightarrow^* (ab)^{j+2}\ aa\ (ba)^t(ab)^{t-j}\\ &\Rightarrow^* (ab)^{j+2}\ aa\ (ba)^{2t-j}\\ &\Rightarrow^* aa\ (ba)^{2t-2j-2}\\ &\rightarrow^* (ba)^{2t-2j-2}\ aa,\end{aligned}$$

obtaining a word and its reversal. Note that also here it was essential that rose R_2^1 was used instead of rose R_2.

(c) The last possibility is that $\diamond_b > \diamond_a$ and $\diamond_{b-k} < \diamond_{a-k}$. Then $\alpha(a,b) = a^i(ba)^j(ab)^ja^i$ and $\gamma(a,b) = b^s(ab)^t(ba)^tb^s$ for some $i,s \geq 1$ and $j,t \geq 0$. We have

$$\begin{aligned} u &\Rightarrow^* a^i (ba)^j (ab)^j a^i \; bbaa \; b^s (ab)^t (ba)^t b^s \\ &\Rightarrow^* a^{i-1} (ab)^{2j} a^{i-1} \; aa \; bbaa \; bbb^{s-1} (ab)^{2t} b^{s-1} \\ &\Rightarrow^* a^{i-1} b^{s-1} (ab)^{2j} aba \; baba \; b(ab)^{2t} b^{s-1} a^{i-1} \\ &\Rightarrow^* (ab)^{2j+2t+4} \\ &\to^* (ba)^{2j+2t+4}. \end{aligned}$$

The last two words are reversals of each other.

Case 4: None of the cases 1–3 apply

Because Case 3 does not apply, directions a and b do not appear on the same rose segment; because Case 1 does not apply the directions appear on unique rose segments, with the antiparallel directions then appearing on the opposite rose segments. We can orient rhombus S so that b appears in segment B and $\overleftarrow{a}$ in segment D. Then $\beta(a,b) = bb$ and $\delta(a,b) = \overleftarrow{a}\overleftarrow{a}$. Now $\alpha(a,b)$ and $\gamma(a,b)$ are palindromes containing only letters a and b, and only letters $\overleftarrow{a}$ and b, respectively. Then

$$w = bb \; \gamma(a,b) \; \overleftarrow{a}\overleftarrow{a} = \beta(a,b)\gamma(a,b)\delta(a,b)$$

contains only letters $\overleftarrow{a}$ and b.

Depending on whether $\diamond_a > \diamond_b$ or $\diamond_a < \diamond_b$ we have either $\alpha(a,b) = b^i (ab)^j (ba)^j b^i$ or $\alpha(a,b) = a^i (ba)^j (ab)^j a^i$, for some $i \geq 1$ and $j \geq 0$. In the first case,

$$\begin{aligned} u &= b^i (ab)^j (ba)^j b^i \; w \\ &\Rightarrow^* b^i (ba)^{2j} \; w \; b^i \\ &\Rightarrow^* (ba)^{2j} \; w, \end{aligned}$$

and also in the second case

$$\begin{aligned} u &= a^i (ba)^j (ab)^j a^i \; w \\ &\Rightarrow^* (ba)^{2j} a^i \; w \; \overleftarrow{a}^i \\ &\Rightarrow^* (ba)^{2j} a^i \overleftarrow{a}^i \; w \\ &\Rightarrow^* (ba)^{2j} \; w. \end{aligned}$$

We see that in any case, $u \Rightarrow^* (ba)^{2j} \; bb \; \gamma(a,b) \; \overleftarrow{a}\overleftarrow{a}$.

Analogously, depending on whether $\diamond_{a-k} > \diamond_{b-k}$ or $\diamond_{a-k} < \diamond_{b-k}$ we have either $\gamma(a,b) = b^s (\overleftarrow{a}b)^t (b\overleftarrow{a})^t b^s$ or $\gamma(a,b) = \overleftarrow{a}^s (b\overleftarrow{a})^t (\overleftarrow{a}b)^t \overleftarrow{a}^s$, for some $s \geq 1$ and $t \geq 0$. In the first case,

$$\begin{aligned} u &\Rightarrow^* (ba)^{2j} \; bb \; b^s (\overleftarrow{a}b)^t (b\overleftarrow{a})^t b^s \; \overleftarrow{a}\overleftarrow{a} \\ &\Rightarrow^* b^s \; (ba)^{2j} \; (b\overleftarrow{a})^{2t+2} \; b^s \\ &\Rightarrow^* (ba)^{2j} \; (b\overleftarrow{a})^{2t+2}. \end{aligned}$$

According to Lemma 3, the crossing condition is satisfied. In the second case,

$$\begin{aligned} u &\Rightarrow^* (ba)^{2j}\ bb\ \overleftarrow{a}^s (b\overleftarrow{a})^t (\overleftarrow{a}b)^t \overleftarrow{a}^s\ \overleftarrow{a}\overleftarrow{a} \\ &\Rightarrow^* (ba)^{2j}\ \overleftarrow{a}^s\ (b\overleftarrow{a})^{2t+2}\ \overleftarrow{a}^s \\ &\Rightarrow^* a^s\ (ba)^{2j}\ \overleftarrow{a}^s\ (b\overleftarrow{a})^{2t+2} \\ &\Rightarrow^* (ba)^{2j}\ a^s\ \overleftarrow{a}^s\ (b\overleftarrow{a})^{2t+2} \\ &\Rightarrow^* (ba)^{2j}\ (b\overleftarrow{a})^{2t+2}, \end{aligned}$$

so again Lemma 3 applies.

This completes the case analysis. All possible cases were covered, and the crossing condition was confirmed in each case. This completes the proof of Theorem 1. □

6 Conclusions

We have demonstrated primitive substitutions on unit rhombuses that generate substitution tilings with $2n$-fold rotational symmetry for all n. The obtained tilings are uniformly recurrent, *i.e.*, quasiperiodic. The proof was based on a rewrite system on the proposed boundaries of the enlarged rhombuses to check that the interior can be properly tiled.

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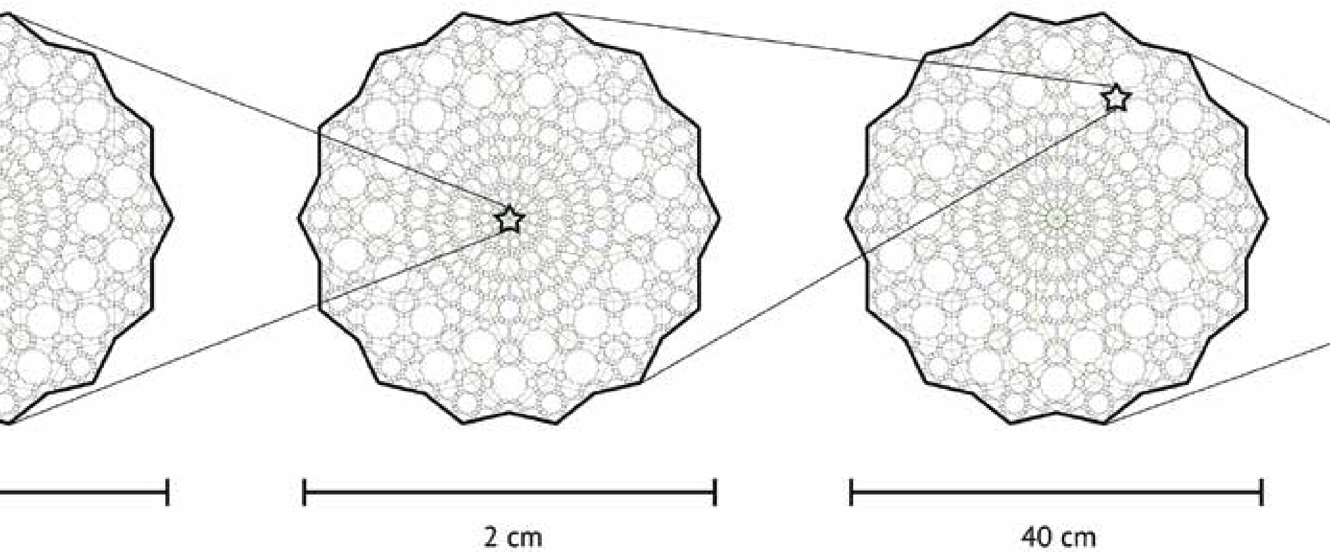
To make Appendix C more tangible, I add this short explanation. The Sub Rosa pattern for *n* = 7 depicted in the separate broadsheet is *c.* 40 cm in diameter, expressed in round numbers.

The Sub Rosa is a self-similar pattern; that is, its details are miniature copies of the whole. This double-page spread illustrates this property. A chain of Type 2 Roses is shown here, each Rose drawn in a different scale and each connecting to a different location in the pattern of the Rose on its right side.

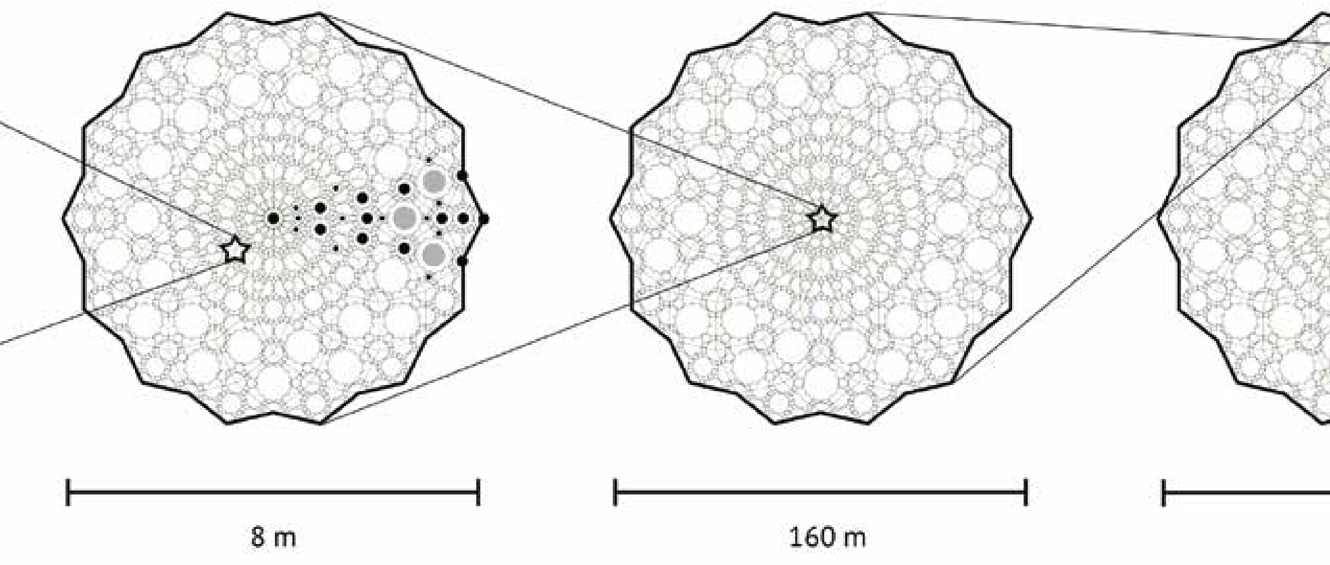
Moving left to right, each new image depicts a part of the environment that surrounds the Rose on the left side. The environment of the 2 cm Rose, for example, is a similar 40 cm Rose, which in turn is located in a similar 8 m Rose, which in turn is located in a similar 160 m Rose, which in turn is located in a similar 3.2 km Rose, etc. As the Sub Rosa scaling factor for *n* = 7 is *c.* 20, there is a 20-fold increase in the scale for each step when moving left to right.



Moving right to left, each new image depicts one small detail from the pattern on the right side. The 8 m Rose, for example, depicts a detail from the 160 m Rose. The 40 cm Rose, in turn, depicts a detail from the 8 m Rose, and so on. Moving right to left, there is a 20-fold decrease in scale for each step.

Inside the 8 m Rose, I have marked a few Roses of Types 1, 2 and 3, found in this particular $n = 7$ Sub Rosa pattern. The very tiny black dots indicate Type 1 Roses, and the slightly larger black dots denote Type 2 Roses. A few Type 3 Roses are marked with grey circles.

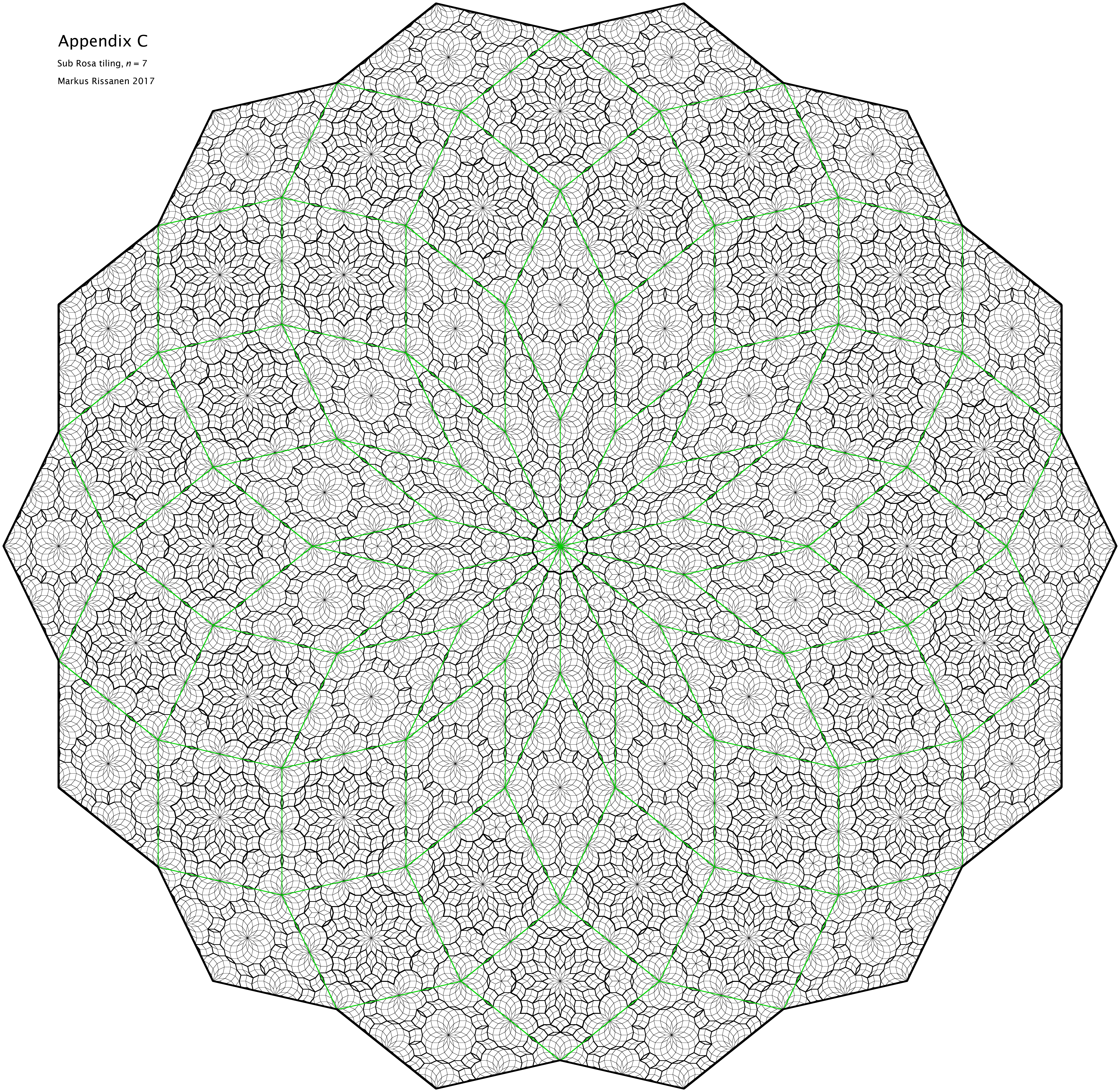
Just as there are few individual Roses marked in the 8 m Rose, only a few parts of the whole pattern are shown in these images, as the unit rhombuses simply cannot be depicted at this size. For a 40 cm pattern complete with all of the unit rhombuses, see Appendix C.



Appendix C

Sub Rosa tiling, $n = 7$

Markus Rissanen 2017



Do basic geometrical forms exist in nature, or are they merely products of the human mind? What roles do the three simple forms of classic Euclidean geometry – the circle, square and triangle – play in the cultural history of forms? Have we reached the limits of describing nature and its functions with basic forms? How do rhombuses help us to decipher the laws of rotational symmetry? This thesis is an interdisciplinary study of basic forms and nature. It combines artistic and scientific modes of research and consists of artistic productions and the cultural-historical study of forms as well as mathematical explanations.

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UNIVERSITY OF THE ARTS HELSINKI

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Markus Rissanen

Basic Forms and Nature

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